Grad R.R.S

AMERICAN
Journal of Mathematics

FRANK MORLEY, EDITOR
A. COHEN, ASSISTANT EDITOR
WITH THE COOPERATION OF
CHARLOTTE A. SCOTT, A. B. COBLE
AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

Πραγμάτων ἔλεγχος οὐ βλεπομένων

VOLUME XLIV, NUMBER 2

BALTIMORE: THE JOHNS HOPKINS PRESS

LEMCKE & BUECHNER, *New York.*
E. STEIGER & CO., *New York.*
G. E. STECHERT & CO., *New York.*
WILLIAM WESLEY & SON, *London.*
A. HERMANN, *Paris.*
ARTHUR F. BIRD, *London.*

APRIL, 1922

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing as special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

CONTENTS

A Primary Classification of Projective Transformations in Function Space. By L. L. DINES, ... 87

A General Theory of Limits. By E. H. MOORE and SMITH, ...

Substitution Groups Whose Cycles of the Same Order Contain a Given Number of Letters. By G. A. MILLER, . . 122

Boundary Value and Expansion Problems: Oscillation, Comparison and Expansion Theorems. By R. D. CARMICHAEL, 129

On a Theorem in General Analysis and the Interrelations of Eight Fundamental Properties of Classes of Functions. By E. W. CHITTENDEN, ... 153

THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL is $6.00 a volume (foreign postage, 50 cents); single numbers, $1.75. A few complete sets of the JOURNAL remain on sale.

It is requested that all editorial communications be addressed to the Editors of the AMERICAN JOURNAL OF MATHEMATICS, and all business or financial communications to The Johns Hopkins Press, Baltimore, Md., U. S. A.

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

A PRIMARY CLASSIFICATION OF PROJECTIVE TRANSFORMATIONS IN FUNCTION SPACE.*

By L. L. Dines.

In an earlier paper† we have defined and studied the projective functional transformation

$$\phi'(x) = \frac{\alpha(x) + \beta(x)\phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy}{\delta + \int_0^1 \epsilon(y)\phi(y)dy}, \tag{1}$$

represented symbolically by the array of coefficients

$$\begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}.$$

Obviously the transformation (1) includes as a special case the Fredholm transformation

$$\phi'(x) = \phi(x) + \int_0^1 \gamma(x, y)\phi(y)dy.$$

It also includes other transformations which may properly be said to be *projectively equivalent* to Fredholm transformations. One of the objects of the present paper is to characterize this class of transformations.

Such projective transformations as do not fall into the above class are themselves divided into three classes on the basis of projective equivalence. The classification is accomplished by a consideration of invariant elements of two kinds: *points* and *lineoids*, the lineoid being the natural dual of the point in function space.

§ 1. Homogeneous Coördinates.

By virtue of its form, (1) transforms every point $\phi(x)$ of function space $\mathfrak{C}_1$ into a point $\phi'(x)$ of the same space, with the exception of those points defined by the equation

$$\delta + \int_0^1 \epsilon(y)\phi(y)dy = 0.$$

With a view to attaining a domain which is closed under the projective transformation we introduce homogeneous coördinates. To this end, we make in (1) the substitutions

$$\phi(x) = \omega_1(x)/\omega_2, \qquad \phi'(x) = \omega_1'(x)/\omega_2', \tag{2}$$

* Read under slightly different title before the American Mathematical Society, September 3, 1919.

† *Transactions of the American Mathematical Society*, Vol. 20, pages 45–65. This paper will be referred to as *Proj. Trans.*

with the understanding that

$$\omega_1(x) \text{ and } \omega_1'(x) \text{ belong to } \mathfrak{C}_1,$$
$$\omega_2 \quad \text{and} \quad \omega_2' \quad \text{belong to } \mathfrak{C}_0',$$

where $\mathfrak{C}_0'$ denotes the class of real constants with 0 excluded.

The resulting transformation is equivalent to the linear *homogeneous* transformation

$$(3) \qquad \begin{aligned} r\omega_1'(x) &= \beta(x)\omega_1(x) + \int_0^1 \gamma(xy)\omega_1(y)dy + \alpha(x)\omega_2 \\ r\omega_2' &= \int_0^1 \epsilon(y)\omega_1(y)dy + \delta\omega_2 \end{aligned}$$

where r is a factor of proportionality.

This homogeneous transformation operates upon the binary elements

$$(\omega_1(x), \omega_2)$$

of the composite class $\mathfrak{C}_1\mathfrak{C}_0$, transforming each into a binary element

$$(r\omega_1'(x), r\omega_2')$$

of the same class. If ω_2 and ω_2' are restricted to be different from zero, (3) is equivalent to (1) in the light of the substitution (2). Nothing inherent in the form of the homogeneous transformation (3) however demands this restriction.

Following the suggestion dictated by analogy with ordinary projective geometry, we will interpret the binary element $(\omega_1(x), \omega_2)$ in which *either one* of the constituent elements is different from zero, as the homogeneous coördinates of a point in function space, it being understood that $(\omega_1(x), \omega_2)$ and $(\omega_1'(x), \omega_2')$ represent the same point, if and only if there is a constant r such that $\omega_1 = r\omega_1'$ and $\omega_2 = r\omega_2'$.

The class

$$\overline{\mathfrak{C}_1\mathfrak{C}_0} \equiv [\textit{all points } (\omega_1(x), \omega_2')]$$

will constitute the *projective function space,* and the transformation (3) the projective functional transformation in homogeneous coördinates.

The results obtained in *Proj. Trans.* can easily be translated into terms of homogeneous coördinates. For example the inverse of (3), in case the determinant B is different from zero, is

$$(3') \qquad \begin{aligned} r'\omega_1(x) &= \beta'(x)\omega_1'(x) + \int_0^1 \gamma'(xy)\omega_1'(y)dy + \alpha'(x)\omega_2', \\ r'\omega_2 &= \int_0^1 \epsilon'(y)\omega_1'(y)dy + \delta'\omega_2', \end{aligned}$$

where the coefficients α', β', γ', δ', ϵ' have the same definitions as in *Proj. Trans.* (8).

§ 2. THE LINEOID, AND DUALITY.

The class of points $(\omega_1(x), \omega_2)$ satisfying an equation of form

$$\int_0^1 v_1(y)\omega_1(y)dy + v_2\omega_2 = 0 \tag{4}$$

where v_1 belongs to $\mathfrak{C}_1$ and v_2 to $\mathfrak{C}_0$ and one at least is different from zero, will be called a *lineoid* in function space.

Since the lineoid (4) is characterized by the pair of coefficients

$$(v_1(y), v_2),$$

this pair of coefficients will serve as coördinates of the lineoid, evidently homogeneous coördinates, since the lineoids $(v_1(y), v_2)$ and $(v_1'(y), v_2')$ coincide if and only if there is a constant r such that $v_1(y) = rv_1'(y)$ and $v_2 = rv_2'$.

The equation (4) expresses the dualistic relation of conjointness between a point $(\omega_1(x), \omega_2)$ and a lineoid $(v_1(y), v_2)$ which simultaneously satisfy it.

Suppose now that every point of a given lineoid (4) be subjected to a projective transformation (3). What will be the locus of the transformed points $(\omega_1'(x), \omega_2')$? To ascertain this we introduce the value of $(\omega_1(x), \omega_2)$ in terms of $(\omega_1'(x), \omega_2')$ from (3′) into (4), and obtain an equation which reduces to the form

$$\int_0^1 v_1'(y)\omega_1'(y) + v_2'\omega_2' = 0,$$

where

$$\begin{aligned} rv_1'(y) &= v_1(y)\beta'(y) + \int_0^1 v_1(x)\gamma'(xy)dx + v_2\epsilon'(y), \\ rv_2' &= \int_0^1 v_1(x)\alpha'(x)dx + v_2\delta'. \end{aligned} \tag{5}$$

This is by definition the equation of a lineoid. Hence, *the projective transformation* (3) *transforms every lineoid into a lineoid.* More explicitly, *the lineoid* $(v_1(y), v_2)$ *is transformed by the projective transformation* (3) *into a lineoid* $(v_1'(y), v_2')$ *whose coördinates are given by* (5). Also, *a point and a lineoid which are conjoint are transformed by a projective transformation into a point and a lineoid which are conjoint.*

§ 3. EXPRESSION OF THE PROJECTIVE TRANSFORMATION IN THE FORM OF A FREDHOLM TRANSFORMATION.

It has recently been pointed out by Hildebrandt* that the transformation (1) can be expressed in the form of a Fredholm transformation, operative upon functions on a suitably defined range. The procedure used in this section is essentially that suggested by Hildebrandt.

By our conventions in *Proj. Trans.*, the range of the argument in the functions of $\mathfrak{C}_1$ is the class of real numbers from 0 to 1, inclusive. Let us

* *Bulletin of the American Mathematical Society*, Vol. 26 (1920), page 400.

denote this range by $\mathfrak{I}_1$; that is

$$\mathfrak{I}_1 \equiv [\textit{all real numbers from } 0 \textit{ to } 1, \textit{ inclusive}].$$

Let us likewise define

$$\mathfrak{I}_2 \equiv [\textit{all real numbers from } 1 \textit{ to } 2, \textit{ including } 2],$$
$$\mathfrak{I}_1 + \mathfrak{I}_2 \equiv [\textit{all real numbers from } 0 \textit{ to } 2, \textit{ inclusive}].$$

By composition of these ranges we get the following composite ranges (each represented geometrically by the points of a square):

$$\mathfrak{I}_1\mathfrak{I}_1, \quad \mathfrak{I}_1\mathfrak{I}_2, \quad \mathfrak{I}_2\mathfrak{I}_1, \quad \mathfrak{I}_2\mathfrak{I}_2, \quad (\mathfrak{I}_1 + \mathfrak{I}_2)(\mathfrak{I}_1 + \mathfrak{I}_2).$$

Now let us suppose the binary elements $(\omega_1(x), \omega_2)$ and $(\omega_1'(x), \omega_2')$ to be replaced respectively by the binary elements $(\omega_1(x), \omega_2(x))$ and $(\omega_1'(x), \omega_2'(x))$, with the understanding that $\omega_2(x)$ and $\omega_2'(x)$ are constant functions having the definitions:

$$\omega_2(x) = \omega_2, \qquad \omega_2'(x) = \omega_2', \qquad x \text{ on } \mathfrak{I}_2.$$

Then the equations of (3) are equivalent to

$$(6) \qquad \begin{aligned} r\omega_1'(x) &= \beta(x)\omega_1(x) + \int_0^1 \gamma(xy)\omega_1(y)dy + \int_1^2 \alpha(x)\omega_2(y)dy, \\ r\omega_2'(x) &= \omega_2(x) + \int_0^1 \epsilon(y)\omega_1(y)dy + \int_1^2 (\delta - 1)\omega_2(y)dy, \end{aligned}$$

and by the classic method of Fredholm* for reducing a system of integral equation to a single equation, this pair of equations can be represented by

$$(7) \qquad r\rho(x)\omega'(x) = \omega(x) + \int_0^2 \kappa(xy)\omega(y)dy, \qquad x \text{ on } \mathfrak{I}_1 + \mathfrak{I}_2,$$

where

$$\omega(x) = \begin{cases} \omega_1(x) & \text{on } \mathfrak{I}_1 \\ \omega_2 & \text{on } \mathfrak{I}_2 \end{cases}$$

$$\omega'(x) = \begin{cases} \omega_1'(x) & \text{on } \mathfrak{I}_1 \\ \omega_2' & \text{on } \mathfrak{I}_2 \end{cases}$$

$$\rho(x) = \begin{cases} 1/\beta(x) & \text{on } \mathfrak{I}_1 \\ 1 & \text{on } \mathfrak{I}_2 \end{cases}$$

$$\kappa(x, y) = \begin{cases} \gamma(x, y)/\beta(x) & \text{on } \mathfrak{I}_1\mathfrak{I}_1 \\ \alpha(x)/\beta(x) & \text{on } \mathfrak{I}_1\mathfrak{I}_2 \\ \epsilon(y) & \text{on } \mathfrak{I}_2\mathfrak{I}_1 \\ \delta - 1 & \text{on } \mathfrak{I}_2\mathfrak{I}_2 \end{cases}$$

The transformation (7) is an ordinary Fredholm transformation, transforming a function $\omega(x)$ on $\mathfrak{I}_1 + \mathfrak{I}_2$ into a function $r\rho(x)\omega'(x)$ on the same interval. The kernel $\kappa(xy)$ is continuous on the square $(\mathfrak{I}_1 + \mathfrak{I}_2)(\mathfrak{I}_1 + \mathfrak{I}_2)$ except for the lines of possible discontinuity $x = 1$ and $y = 1$.

In a similar manner, the lineoid transformation (5) can be replaced by

$$(7') \qquad r\rho'(y)v'(y) = v(y) + \int_0^2 v(x)\kappa'(x, y)dx, \qquad y \text{ on } \mathfrak{I}_1 + \mathfrak{I}_2,$$

* Acta mathematica, Vol. 27.

where

$$v(y) = \begin{cases} v_1(y) & on \mathfrak{I}_1 \\ v_2 & on\ \mathfrak{I}_2 \end{cases} \qquad v'(y) = \begin{cases} v_1(y) & on\ \mathfrak{I}_1 \\ v_2' & on\ \mathfrak{I}_2 \end{cases} \qquad \rho'(y) = \begin{cases} 1/\beta'(y) & on\ \mathfrak{I}_1 \\ 1 & on\ \mathfrak{I}_2 \end{cases}$$

$$\kappa'(x, y) = \begin{cases} \gamma'(x, y)/\beta'(y) & on\ \mathfrak{I}_1\mathfrak{I}_1 \\ \epsilon'(y)/\beta'(y) & on\ \mathfrak{I}_2\mathfrak{I}_1 \\ \alpha'(x) & on\ \mathfrak{I}_1\mathfrak{I}_2 \\ \delta' - 1 & on\ \mathfrak{I}_2\mathfrak{I}_2 \end{cases}$$

The kernels $\kappa(x, y)$ and $\kappa'(x, y)$ of the transformations (7) and (7′) satisfy the two identities

$$\kappa(x, y) + \kappa'(x, y) + \int_0^2 \kappa(x, z)\kappa'(z, y)dz = 0$$
$$\kappa'(x, y) + \kappa(x, y) + \int_0^2 \kappa'(x, z)\kappa(z, y)dz = 0$$

on the square $(\mathfrak{I}_1 + \mathfrak{I}_2)(\mathfrak{I}_1 + \mathfrak{I}_2)$, as may be verified by use of the relations (9) and (9′) of *Proj. Trans.* That is, they are *reciprocal kernels* in the Fredholm sense.

§ 4. Invariant Points and Lineoids.

We now inquire as to what points, if any, are transformed into themselves by (3), restricting ourselves for the present however to the important special case in which

(8) $$\beta(x) = 1.$$

From (7) which is equivalent to (3), we see that under the condition (8), the equation for such invariant points is

(9) $$(1 - r)\omega(x) + \int_0^2 \kappa(x, y)\omega(y)dy = 0, \qquad x\ on\ \mathfrak{I}_1 + \mathfrak{I}_2.$$

Every non-zero solution $\omega(x)$ of this equation determines a binary element $(\omega_1(x), \omega_2(x))$ in which

$$\omega_1(x) = \omega(x) \qquad on\ \mathfrak{I}_1,$$
$$\omega_2(x) = \omega(x) \qquad on\ \mathfrak{I}_2.$$

If $\omega_2(x)$ is constant, this binary element represents an invariant point. And $\omega_2(x)$ cannot be other than a constant except in the single case $r = 1$, as will be evident upon consideration of the content of equation (9) when x is on $\mathfrak{I}_2$, namely

(10) $$(1 - r)\omega_2(x) + \int_0^1 \epsilon(y)\omega_1(y)dy + \int_0^2 (\delta - 1)\omega_2(y)dy = 0.$$

In case $r = 1$, a solution $\omega = (\omega_1(x), \bar{\omega}_2(x))$ may exist in which $\bar{\omega}_2(x)$ is not constant, but corresponding to every such solution there will be a solution $\omega = (\omega_1(x), \omega_2)$ in which $\omega_2 = \int_0^1 \bar{\omega}_2(y)dy$. This solution will represent an invariant point unless $\omega_1(x) = \omega_2 = 0$.

Equation (9) is an homogeneous Fredholm integral equation containing the parameter r. The non-zero solutions $\omega(x)$ corresponding to values of r different from 1 are the so-called *fundamental functions in x* for the kernel $\kappa(x, y)$. If we denote by the term *asymptotic fundamental functions* those non-zero solutions $\omega(x)$ corresponding to the parameter value $r = 1$, we may state our conclusions in the form

THEOREM I: *If $\beta(x) = 1$, the invariant points $\omega(x) = (\omega_1(x), \omega_2)$ of the transformation* (3) *are precisely the ordinary fundamental functions in x of the kernel $\kappa(x, y)$, together with those asymptotic fundamental functions in x which are constant on the interval $\mathfrak{I}_2$.*

By an analogous consideration of (7′), we obtain

THEOREM I′: *If $\beta(x) = 1$, the invariant lineoids $v(y) = (v_1(y), v_2)$ of the transformation* (3) *are precisely the ordinary fundamental functions in y of the kernel $\kappa'(x, y)$, together with those asymptotic fundamental functions in y which are constant on the interval $\mathfrak{I}_2$.*

In this theorem the invariant lineoids are characterized by means of the reciprocal kernel $\kappa'(x, y)$. They are characterized directly in terms of $\kappa(x, y)$ in the following

COROLLARY: *If $\beta(x) = 1$, the invariant lineoids of the transformation* (3) *are precisely the ordinary fundamental functions in y of the kernel $k(x, y)$, together with those asymptotic fundamental functions which are constant on the interval $\mathfrak{I}_2$.*

To prove this we need only show that the solutions of the two equations

$$(11') \qquad (1 - r')v(y) + \int_0^2 v(x)\kappa'(x, y)dx = 0$$

and

$$(11) \qquad (1 - r)v(y) + \int_0^2 v(x)\kappa(x, y)dx = 0$$

coincide.

Suppose $\bar{v}(y)$ is a solution of (11′). Then for some value of r',

$$(12) \qquad r'\bar{v}(y) = \bar{v}(y) + \int_0^2 \bar{v}(x)\kappa'(x, y)dx.$$

We multiply equation (11) by r' and write it in the form

$$(13) \qquad r'rv(y) = r'v(y) + r'\int_0^2 v(x)\kappa(x, y)dx.$$

Upon substituting in (13) $\bar{v}$ for v in the left member and the value of $r'\bar{v}$ from (12) for $r'v$ in the right member, we obtain after some rearrangement

$$r'r\bar{v}(y) = \bar{v}(y) + \int_0^2 \bar{v}(x)\left[\kappa'(x, y) + \kappa(x, y) + \int_0^2 \kappa'(x, z)\kappa(z, y)dz\right]dx.$$

Since $\kappa(x, y)$ and $\kappa'(x, y)$ are reciprocal kernels this reduces to

$$r'r\bar{v}(y) = \bar{v}(y),$$

which proves that $\bar{v}(y)$ is a solution of (11) corresponding to the parameter value $r = 1/r'$.

In a similar manner it can be shown that each solution of (11) is a solution of (11′).

§ 5. Symmetric Projective Transformations.

The theorems of the preceding section suggest interesting properties of those projective transformations for which the corresponding kernels $\kappa(x, y)$ are symmetric. The most general such transformation is of form

$$r\omega_1'(x) = \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2,$$
$$r\omega_2' = \int_0^1 \alpha(y)\omega_1(y)dy + \delta\omega_2,$$

where $\gamma(x, y) = \gamma(y, x)$.

Transformations of this form will be called *symmetric projective transformations.*

From the well-known theory of integral equations with symmetric kernels,* we have by virtue of the theorems of the preceding section

THEOREM II. *Every symmetric projective transformation has at least one invariant point, and one invariant lineoid, which are not conjoint.*

The existence of the invariant elements follows from the fact that every symmetric kernel has a characteristic value; the non-conjointness from the fact that every such characteristic value is a simple pole of the resolvent, and hence (see Goursat, loc. cit., page 411) for every fundamental function ω in x of the kernel $\kappa(x, y)$ there is a fundamental function v in y such that

$$\int_0^2 v(y)\omega(y)dy \neq 0,$$

that is

$$\int_0^1 v_1(y)\omega_1(y)dy + v_2\omega_2 \neq 0.$$

§ 6. Projective Transformations without Invariant Points and Lineoids.

While, as we have just seen, every symmetric projective transformation admits an invariant point and lineoid, such is not the case with non-symmetric transformations. In this section we present four examples of transformations qualified respectively as follows:

1. Admitting an invariant lineoid, but no invariant point.
2. Admitting an invariant point, but no invariant lineoid.
3. Admitting neither invariant point nor lineoid.
4. Admitting an invariant point and an invariant lineoid, but no pair which are not conjoint.

* See, for instance, Goursat's *Cours d'analyse mathematique,* Vol. III, § 587.

We shall make use of the following

LEMMA:* If the traces

$$A_n = \int_0^2 \int_0^2 \cdots \int_0^2 \kappa(s_1, s_2)\kappa(s_2, s_3) \cdots \kappa(s_n, s_1) ds_1 ds_2 \cdots ds_n$$

of the kernel $\kappa(x, y)$ are zero for all values of n greater than or equal to 3, then $\kappa(x, y)$ admits no characteristic value, and hence no ordinary fundamental function.

We take now any complete† normalized orthogonal system of continuous functions ϕ_n, $n = 1, 2, \cdots$, on the interval $\mathfrak{I}_1$.

From this system which is orthogonal on $\mathfrak{I}_1$ we form a system Φ_n, $n = 0, 1, 2, \cdots$, orthogonal on $\mathfrak{I}_1 + \mathfrak{I}_2$, as follows:

$$\Phi_0 \equiv [0 \text{ on } \mathfrak{I}_1,\ 1 \text{ on } \mathfrak{I}_2],$$
$$\Phi_n \equiv [\phi_n \text{ on } \mathfrak{I}_1,\ 0 \text{ on } \mathfrak{I}_2], \qquad n = 1, 2, \cdots.$$

Our examples will be constructed in terms of the functions $\{\phi_n\}$; and in each case the kernel $\kappa(x, y)$ to which the transformation gives rise will be expansible in terms of the functions $\{\Phi_n\}$.

Example 1. Consider the transformation

$$\begin{pmatrix} 1+\gamma & \alpha \\ 0 & 1 \end{pmatrix} \tag{14}$$

in which

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n \phi_{n+1}(x)\phi_n(y), \qquad \alpha(x) \equiv \phi_1(x),$$

the coefficients c_n being all distinct from zero and so chosen that the series for $\gamma(x, y)$ converges uniformly on the square $\mathfrak{I}_1\mathfrak{I}_1$.

The corresponding kernel $\kappa(x, y)$ can be written in the form

$$\kappa(x, y) \equiv \sum_{n=0}^{\infty} c_n \Phi_{n+1}(x)\Phi_n(y) \quad \text{on } (\mathfrak{I}_1 + \mathfrak{I}_2)(\mathfrak{I}_1 + \mathfrak{I}_2),$$

where $c_0 = 1$.

The traces of this kernel $\kappa(x, y)$ are all zero. Hence by the lemma the kernel admits no singular values, and no ordinary fundamental functions.

Nor does it admit any asymptotic fundamental functions which can furnish invariant points. For the equation

$$\int_0^2 \kappa(x, y)\omega(y)dy = 0$$

is equivalent to

$$\sum_{n=0}^{\infty} c_n \Phi_{n+1}(x) \int_0^2 \Phi_n(y)\omega(y)dy = 0.$$

* Cf. Goursat, loc. cit., page 428.

† An orthogonal system of functions is said to be *complete* if there is no function which is orthogonal to all the functions of the system, and such that the integral of its square is 1. Cf. Goursat, loc. cit., pages 445 and 446.

Multiplying this equation by $\Phi_{k+1}(x)$ and integrating with respect to x, we get, on account of the orthogonality,

$$c_k \int_0^2 \Phi_k(y)\omega(y)dy = 0, \qquad k = 0, 1, 2, \cdots.$$

That is, since $c_k \neq 0$, $\omega(y)$ must be orthogonal to all the functions $\Phi_k(y)$, on the interval $\mathfrak{I}_1 + \mathfrak{I}_2$.

But from the definitions of the Φ's, it is clear that there can be no such function ω which is constant on $\mathfrak{I}_2$. Hence, *the transformation*

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2, \\ r\omega_2' &= \omega_2, \end{aligned}$$

represented by (14), *admits no invariant point. It admits the invariant lineoid* $\omega_2 = 0$.

Example 2. *The transformation*

$$\begin{aligned} r\omega_1'(x) &= \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy, \\ r\omega_2' &= \omega_2 + \int_0^1 \epsilon(y)\omega_1(y)dy, \end{aligned}$$

in which

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n \phi_n(x)\phi_{n+1}(y), \qquad \epsilon(y) \equiv \phi_1(y),$$

has no invariant lineoid, but admits the invariant point (0, 1).

The kernel $\kappa(x, y)$ for this transformation can be written in the form

$$\kappa(x, y) = \sum_{n=0}^{\infty} c_n \Phi_n(x)\Phi_{n+1}(y),$$

from which fact the properties of the transformation follow as in Example 1.

Example 3. Consider the transformation whose coefficients are*

$$\gamma(x, y) \equiv \sum_{n=1}^{\infty} c_n[\phi_{2n+1}(x)\phi_{2n-1}(y) + \phi_{2n}(x)\phi_{2n+2}(y)], \tag{15}$$

$$\alpha(x) \equiv \phi_1(x), \qquad \epsilon(y) \equiv \phi_2(y), \qquad \delta \equiv 1.$$

The corresponding kernel $\kappa(x, y)$ can be written in the form

$$\kappa(x, y) = \Phi_1(x)\Phi_0(y) + \Phi_0(x)\Phi_2(y) \sum_{n=1}^{\infty} c_n[\Phi_{2n+1}(x)\Phi_{2n-1}(y) + \Phi_{2n}(x)\Phi_{2n+2}(y)].$$

All the traces of this kernel are zero; hence it has no ordinary fundamental functions. Nor does it admit asymptotic fundamental functions in either x or y which are constant on $\mathfrak{I}_2$, as may be shown by the method used in Example 1. Hence *the transformation defined by* (15) *admits no invariant element.*

* The form of this transformation was suggested by an example kindly furnished me by Professor E. W. Chittenden.

Example 4. Consider the transformation

$$r\omega_1'(x) = \omega_1(x) + \int_0^1 \gamma(x, y)\omega_1(y)dy + \alpha(x)\omega_2,$$
$$r\omega_2' = \omega_2,$$

where

$$\gamma(x, y) \equiv \sum_{n=2}^{\infty} c_n\phi_{n+1}(x)\phi_n(y), \qquad \alpha(x) \equiv \phi_1(x).$$

The corresponding kernel $\kappa(x, y)$ given by

$$\kappa(x, y) = \Phi_1(x)\Phi_0(y) + \sum_{n=2}^{\infty} c_n\Phi_{n+1}(x)\Phi_n(y)$$

has no ordinary fundamental functions since all of its traces are zero.

The transformation admits a single invariant point given by the asymptotic fundamental function

$$\omega(x) = \Phi_1(x) = (\phi_1(x), 0),$$

while it admits the two invariant lineoids

$$v(y) = \Phi_0(y) = (0, 1), \qquad v(y) = \Phi_2(y) = (\phi_2(y), 0).$$

It can be verified immediately that *the invariant point is on both of the invariant lineoids.*

§ 7. Reduction to Canonical Form.

Two transformations

$$\overline{S} \equiv \begin{pmatrix} \overline{\beta} + \overline{\gamma} & \overline{\alpha} \\ & \overline{\epsilon} & \overline{\delta} \end{pmatrix}, \qquad \overline{\overline{S}} \equiv \begin{pmatrix} \overline{\overline{\beta}} + \overline{\overline{\gamma}} & \overline{\overline{\alpha}} \\ & \overline{\overline{\epsilon}} & \overline{\overline{\delta}} \end{pmatrix}$$

are said to be projectively equivalent if there exists a non-singular projective transformation T with inverse T^{-1}:

$$T \equiv \begin{pmatrix} \beta + \gamma & \alpha \\ & \epsilon & \delta \end{pmatrix}, \qquad T^{-1} \equiv \begin{pmatrix} \beta' + \gamma' & \alpha' \\ & \epsilon' & \delta' \end{pmatrix},$$

such that

$$T\overline{S}T^{-1} = \overline{\overline{S}}. \tag{16}$$

The relation between $\overline{S}$ and $\overline{\overline{S}}$ is evidently a reciprocal one, since from (16) it follows that $T^{-1}\overline{\overline{S}}T = \overline{S}$.

Suppose now that $\overline{S}$ admits the invariant point $\bar{\omega} \equiv (\bar{\omega}_1(x), \bar{\omega}_2)$. Then $\overline{\overline{S}}$ admits as invariant point the transform of $\bar{\omega}$ by T. For from the equation

$$\overline{S}\bar{\omega} = r\bar{\omega},$$

which expresses the fact that $\bar{\omega}$ is invariant under $\overline{S}$, there follows by use

of (16)

$$\begin{aligned}\bar{\bar{S}}(T\bar{\omega}) &= T\bar{S}T^{-1}(T\bar{\omega}) \\ &= T\bar{S}\bar{\omega} \\ &= Tr\bar{\omega} \\ &= r(T\bar{\omega}),\end{aligned}$$

which expresses the fact that $T\bar{\omega}$ is invariant under $\bar{\bar{S}}$.

Likewise, if $\bar{S}$ admits the invariant lineoid $\bar{v}$, then $\bar{\bar{S}}$ admits as invariant lineoid the transform of $\bar{v}$ by T.

It is evident from these remarks that a *necessary* condition for projective equivalence is a one-to-one correspondence between the invariant elements of two transformations. This condition is not sufficient, and we shall not undertake here the general question of sufficient conditions.

We shall show, however, that a projective transformation which admits *any* invariant element is projectively equivalent to a transformation of one of the three forms

$$(17) \qquad \begin{pmatrix} \beta+\gamma & 0 \\ \epsilon & \delta \end{pmatrix}, \quad \begin{pmatrix} \beta+\gamma & \alpha \\ 0 & \delta \end{pmatrix}, \quad \begin{pmatrix} \beta+\gamma & 0 \\ 0 & \delta \end{pmatrix},$$

of which the first admits the invariant point (0, 1), the second admits the invariant lineoid (0, 1), while the third admits both of these invariant elements.

We shall need to make use of the explicit forms of the coefficients of $\bar{\bar{S}}$ in terms of the coefficients of T, $\bar{S}$, and T^{-1} as determined by (16). These can easily be written down by use of the formulas obtained in *Proj. Trans.*, § 5. In terms of the abbreviated notations there used, we have

$$(18) \qquad \begin{aligned}
\bar{\bar{\beta}} &= \beta\bar{\beta}\beta' = \bar{\beta}, \\
\bar{\bar{\gamma}} &= \beta\bar{\beta}\gamma' + \beta\bar{\gamma}\beta' + \gamma\bar{\beta}\beta' + (J\gamma\bar{\gamma})\beta' + \alpha\cdot\bar{\epsilon}\beta' + \beta J\bar{\gamma}\gamma' \\
&\quad + J\gamma\bar{\beta}\gamma' + JJ\gamma\bar{\gamma}\gamma' + \alpha J\bar{\epsilon}\gamma' + \beta J\bar{\alpha}\epsilon' + JJ\gamma\bar{\alpha}\epsilon' + J\alpha\bar{\delta}\epsilon', \\
\bar{\bar{\alpha}} &= \beta(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + J\gamma(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + \alpha(J\bar{\epsilon}\alpha' + \bar{\delta}\delta'), \\
\bar{\bar{\epsilon}} &= (\epsilon\bar{\beta} + J\epsilon\bar{\gamma} + \delta\bar{\epsilon})\beta' + J(\epsilon\bar{\beta} + J\epsilon\bar{\gamma} + \delta\bar{\epsilon})\gamma' + (J\epsilon\bar{\alpha} + \delta\bar{\delta})\epsilon', \\
\bar{\bar{\delta}} &= J\epsilon(\bar{\beta}\alpha' + J\bar{\gamma}\alpha' + \bar{\alpha}\delta') + (J\epsilon\bar{\alpha} + \delta\bar{\delta})\delta'.
\end{aligned}$$

Suppose now that $\bar{S}$ admits the invariant point $(\bar{\omega}_1, \bar{\omega}_2)$, that is

$$(19) \qquad \begin{aligned} r\bar{\omega}_1 &= \bar{\beta}\bar{\omega}_1 + J\bar{\gamma}\bar{\omega}_1 + \bar{\alpha}\bar{\omega}_2, \\ r\bar{\omega}_2 &= \qquad\quad J\bar{\epsilon}\bar{\omega}_1 + \bar{\delta}\bar{\omega}_2. \end{aligned}$$

Then if T be so chosen that

$$(\alpha', \delta') \equiv (\bar{\omega}_1, \bar{\omega}_2),$$

the value of $\bar{\bar{\alpha}}$ as determined by (18) may in view of (19) be written

$$\bar{\bar{\alpha}} = r(\beta\alpha' + J\gamma\alpha' + \alpha\delta').$$

Hence

$$\bar{\bar{\alpha}} = 0,$$

on account of the relations between the coefficients of T and T^{-1} (see (9′) of *Proj. Trans.*, I); and the transformation $\bar{\bar{S}}$ projectively equivalent to $\bar{S}$ is of the form occurring first in (17).

Similarly, it may be seen that if $\bar{S}$ admits the invariant lineoid $(\bar{v}_1, \bar{v}_2)$, and if T be so chosen that

$$(\epsilon, \delta) \equiv (\bar{v}_1, \bar{v}_2),$$

then

$$\bar{\bar{\epsilon}} = 0;$$

that is, the transformation $\bar{\bar{S}}$ is of the form appearing second in (17).

The third transformation in (17) admits the invariant point (0, 1) and the invariant lineoid (0, 1)—two elements which are not conjoint. Hence no transformation can be equivalent to it unless it admits two similarly related invariant elements. Let us then suppose that the transformation $\bar{S}$ admits the invariant point $(\bar{\omega}_1, \bar{\omega}_2)$ and the invariant lineoid $(\bar{v}_1, \bar{v}_2)$, and that these two elements are not conjoint, that is

$$J\bar{\omega}_1\bar{v}_1 + \bar{\omega}_2\bar{v}_2 = 1. \tag{20}$$

We shall show that under these conditions, T may be so chosen that $\bar{\bar{S}}$ is of the form occurring third in (17). For this purpose it will be sufficient, in view of the two cases just preceding, to exhibit a transformation T with inverse T^{-1} satisfying the *two* conditions

$$(\epsilon, \delta) \equiv (\bar{v}_1, \bar{v}_2), \qquad (\alpha', \delta') \equiv (\bar{\omega}_1, \bar{\omega}_2).$$

We may on account of (20) and the homogeneity of coördinates assume without loss of generality that *if* $J\bar{\omega}_1\bar{v}_1 = 0$, *then* $\bar{\omega}_2 = \bar{v}_2 = 1$. This will simplify the discussion.

Consider the transformation

$$T = \begin{pmatrix} \beta + \gamma & \alpha \\ \epsilon & \delta \end{pmatrix}$$

in which

$$\beta \equiv 1, \qquad \gamma \equiv c\bar{\omega}_1 \cdot \bar{v}_1, \qquad \alpha \equiv -\bar{\omega}_1, \qquad \epsilon \equiv \bar{v}_1, \qquad \delta \equiv \bar{v}_2, \tag{21}$$

c being a constant to be determined.

The coefficients α', δ' of the inverse transformation T^{-1} are given by the formulas*

$$\alpha'(x) = A(x)/B, \qquad \delta' = D/B,$$

* *Proj. Trans.*, (8).

where for the particular transformation T defined by (21)

$$A(x) = \bar{\omega}_1(x), \qquad D = 1 + cJ\bar{\omega}_1\bar{v}_1,$$
$$B = \bar{v}_2(1 + cJ\bar{\omega}_1 v_1) + J\bar{\omega}_1\bar{v}_1.$$

We now choose c so that

$$D = 1 + cJ\bar{\omega}_1\bar{v}_1 = \bar{\omega}_2.$$

This can obviously be done if $J\bar{\omega}_1\bar{v}_1 \neq 0$, while if $J\bar{\omega}_1\bar{v}_1 = 0$, then $D = 1 = \bar{\omega}_2$ by our assumption. With this choice of c, B becomes by (20) equal to unity. Hence

$$\alpha'(x) = \bar{\omega}_1(x), \qquad \delta' = \bar{\omega}_2,$$

and the transformation defined by (21) possesses the desired properties.

Our results may be stated in the

THEOREM IV: *A projective transformation is projectively equivalent to transformations of the following forms, under the conditions respectively specified, and only under those conditions:*

$\begin{pmatrix} \beta+\gamma & 0 \\ \epsilon & \delta \end{pmatrix}$ *if it admits an invariant point,*

$\begin{pmatrix} \beta+\gamma & \alpha \\ 0 & \delta \end{pmatrix}$ *if it admits an invariant lineoid,*

$\begin{pmatrix} \beta+\gamma & 0 \\ 0 & \delta \end{pmatrix}$ *if it admits an invariant lineoid and an invariant point not on it.*

Upon the basis of this theorem, we are able to classify all projective transformations under the following four types:

Type I, Transformations admitting an invariant lineoid and an invariant point not on it; projectively equivalent to the form

I. $$\begin{pmatrix} \beta+\gamma & 0 \\ 0 & \delta \end{pmatrix}.$$

Type II, Transformations admitting an invariant point, but not belonging to Type I; projectively equivalent to the form

II. $$\begin{pmatrix} \beta+\gamma & 0 \\ \epsilon & \delta \end{pmatrix},$$

but not to the form I.

Type III, Transformations admitting an invariant lineoid, but not belonging to Type I; projectively equivalent to the form

III. $$\begin{pmatrix} \beta+\gamma & \alpha \\ 0 & \delta \end{pmatrix},$$

but not to the form I.

Type IV, Transformations admitting no invariant element; projectively equivalent to no transformation of form I, II, or III.

Each of the four types defines a class of transformations which contains at least one member. Types I and IV define classes which do not overlap each other or either of the remaining classes. The classes defined by Types II and III do overlap, as is shown by Example 4, of § 6.

§ 8. Simplification of the Type Forms by Factorization.

From the first equation of (18) it is clear that the coefficient $\bar{\beta}$ of a transformation $\bar{S}$ is unchanged when $\bar{S}$ undergoes any projective transformation whatever. Any projective transformation may however be *factored* into two transformations of which one is of a very simple type, and the other has the β coefficient equal to unity. In fact it may be verified immediately that

$$\begin{pmatrix} \beta+\gamma & \alpha \\ \epsilon & \delta \end{pmatrix} = \begin{pmatrix} \beta+0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\bar{\gamma} & \bar{\alpha} \\ \epsilon & \delta \end{pmatrix}$$

where $\bar{\gamma}(x, y) = \gamma(x, y)/\beta(x)$, and $\bar{\alpha}(x) = \alpha(x)/\beta(x)$. If $\delta \neq 0$, the form of the transformation may be even further simplified, as follows:

$$\begin{pmatrix} \beta+\gamma & \alpha \\ \epsilon & \delta \end{pmatrix} = \begin{pmatrix} \beta+0 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1+\bar{\gamma} & \bar{\alpha} \\ \bar{\epsilon} & 1 \end{pmatrix}. \tag{22}$$

The transformation forming the first factor on the right side of (22), written in full in non-homogeneous coördinates, is

$$\phi'(x) = \frac{\beta(x)}{\delta}\phi(x),$$

that is, it consists in multiplication by an ordinary function. If we agree to call such a transformation a *functional multiplication*, we may state

THEOREM V: *A non-singular projective transformation of Type I, II, or III, is projectively equivalent to the product of a functional multiplication and a transformation of form*

$$\begin{pmatrix} 1+\gamma & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1+\gamma & \alpha \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 1+\gamma & 0 \\ \epsilon & 1 \end{pmatrix},$$

respectively.

For δ cannot be zero when the transformation in question is reduced to its type form, since a non-singular transformation cannot be projectively equivalent to a singular transformation.

§ 9. FREDHOLM TRANSFORMATIONS.

According to Theorem V, every non-singular transformation of Type I is projectively equivalent to the product of a functional multiplication and a transformation of form

$$\begin{pmatrix} 1+\gamma & 0 \\ 0 & 1 \end{pmatrix}.$$

This latter is, in non-homogeneous coördinates, the ordinary Fredholm transformation

$$\phi' = \phi + J\gamma\phi.$$

Conversely, if a projective transformation is projectively equivalent to the product of a functional multiplication and an ordinary Fredholm transformation, it must admit an invariant lineoid and an invariant point not on it. Hence

THEOREM VI: *A necessary and sufficient condition that a non-singular projective transformation be projectively equivalent to the product of a functional multiplication and an ordinary Fredholm transformation is that it admit an invariant lineoid and an invariant point which are not conjoint.*

COROLLARY: *Every non-singular symmetric projective transformation is projectively equivalent to an ordinary Fredholm transformation and multiplication by a constant.*

UNIVERSITY OF SASKATCHEWAN.

A GENERAL THEORY OF LIMITS.

By E. H. Moore and H. L. Smith.

Introduction.

I. *As to Notions.*—In this paper we investigate a simple general limit of which, as will appear in § 5, the various classical limits of analysis are actually instances.* The general limit in question is an obvious generalization of the following two limits:

1. An infinite sequence $\{a_n\}$ of real or (ordinary) complex numbers a_n $(n = 1, 2, \cdots)$ converges to a number a as a limit, in notation: $L_{n\to\infty} a_n = a$,—as clearly defined about a century ago,—in case for every positive number e there exists a positive integer n_e of such a nature that for every integer $n \geqq n_e$ it is true that in absolute value $a_n - a$ is at most e. Here the numerical sequence $\{a_n\}$ may be considered as a numerically valued function $\alpha \equiv (a_n \mid n)$ of the positive integer n (or on the range $[n]$ of positive integers n), viz., $\alpha(n) \equiv a_n$ for every n.

2. Relative to a general (i.e., any particular) class $\mathfrak{Q} \equiv [q]$ of general elements q and the class $\mathfrak{S} = [s]$ of all finite classes s of elements q, a numerically valued function $\alpha \equiv (\alpha(s) \mid s)$ on the range S converges to a number a as limit, in notation: $L_s\alpha(s) = a$, in case for every positive number e there exists a class s_e of such a nature that for every class s including s_e it is true that in absolute value $\alpha(s) - a$ is at most e. The limit (2) belongs to General Analysis, i.e., to that doctrine of analysis in which a general class, here $\mathfrak{Q}$, plays a fundamental rôle. The limit (2), introduced† in 1915 by the senior author, plays a central rôle in his second theory‡ of Linear Integral Equations in General Analysis. This general theory has as notable instances: (*a*) Hilbert's theory of limited quadratic forms in a denumerable infinitude of variables; here the Hilbert space $[\alpha]$ of infinite sequences $\alpha \equiv (\alpha(n) \mid n)$ of real numbers with convergent $\Sigma_n\alpha(n)\alpha(n)$ plays a central rôle; (*b*) an analogous theory in which the corresponding rôle

* The only other attempt in this direction is by Dimitry-Kryjanowsky, *Nouvelles Annales de Mathématiques*, Ser. 4, Vol. 14 (1914), pp. 49–64. In this paper it is shown how all the classical limits, by means of a transformation, may be reduced to a certain canonical form. The theory is not as general as the present one; indeed it does not include the limit 2) below.

† E. H. Moore, "Definition of Limit in General Integral Analysis," *Proceedings of the National Academy of Sciences*, Vol. 1 (1915), pp. 628–632.

‡ (Addition of Sept. 19, 1922.) The basis of this theory is included in his paper, "On Power Series in General Analysis," *Festschrift David Hilbert zu seinem sechzigsten Geburtstag*, pp. 355–364, Berlin, 1922, and *Mathematische Annalen*, vol. 86, pp. 30–39 (1922).

is played by the Hellinger space $[\alpha]$ of real-valued functions $\alpha \equiv (\alpha(x)\,|\,x)$ of the real variable x on the interval $(0 \leqq x \leqq 1)$ with $\alpha(0) = 0$ and $\int_0^1 \frac{d\alpha(x)d\alpha(x)}{d\beta(x)}$ convergent, β being a fixed monotone increasing function of x; (*c*) various instances involving integration over function-spaces.

It is obvious that the limits (1), (2) are in close analogy. In each case a numerically valued function $\alpha = (\alpha(p)\,|\,p)$ on a certain range $\mathfrak{P} \equiv [p]$ is said to converge, under certain conditions, to a number a as limit; in (1) $\mathfrak{P} \equiv [n]$, in (2) $\mathfrak{P} \equiv \mathfrak{S} \equiv [s]$; moreover, notations apart, the conditions in (1) become the conditions in (2) on replacing the relation $\geqq$, between two positive integers $(n \geqq n_e)$, by the relation *including* or $\supset$, between two classes $(s \supset s_e)$. Thus the authors were led independently to the general limit (3) of which the limits (1), (2) are special instances.

3. Consider a *general class* $\mathfrak{P} \equiv [p]$ of members or *elements* p and a *binary relation* R on the class $\mathfrak{P}$; according as an element p_1 is or is not in the relation R to an element p_2 write $p_1 R p_2$ or $p_1 \bar{}\ R p_2$. Then we say that a *numerically valued function* $\alpha \equiv (\alpha(p)\,|\,p)$* *on the range* $\mathfrak{P}$ *converges* (*with respect to the relation* R) *to a number a as limit,* in notation:

$$\mathrm{L}_p\alpha(p) = a \qquad \text{or} \qquad \mathrm{L}\alpha = a,$$

or, with the relation R in evidence,

$$\mathrm{L}_{p\mathrm{R}}\alpha(p) = a \qquad \text{or} \qquad \mathrm{L}_{\mathrm{R}}\alpha = a,$$

in case for every positive number e there exists an element p_e *of such a nature that for every* $p\mathrm{R}p_e$ (i.e., for every element p in the R relation to p_e) *it is true that in absolute value* $\alpha(p) - a$ *is at most e.*†

In order that the theory of the general limit (3) may include the principal parts of the theories of the limits (1), (2), we impose upon the relation R two conditions obviously satisfied in (1), (2), viz., the conditions: 1) R is *transitive* (R^T),—if p_1 is in the R relation to p_2 and p_2 is in the R relation to p_3, then p_1 is in the R relation to p_3; 2) R has the *composition property* (R^C),—for every two (not necessarily distinct) elements p_1p_2 there exists an element $p_3\mathrm{R}(p_1, p_2)$, that is, an element p_3 in the R relation to each of the elements p_1p_2.

Thus, as *fundamental system* Σ *of notions* for the general limit (3), we have the system:

$$\Sigma \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\mathrm{on}\mathfrak{P}\mathfrak{P}\cdot T C});$$

* Throughout the paper α will denote a numerically valued function on the range, not restricted to be single-valued unless so stated.

† If α is not single-valued, this inequality is understood to hold for all determinations of $\alpha(p)$; unless otherwise stated similar understandings will hold in the case of all inequalities involving multiply-valued functions.

that is, the class $\mathfrak{A} \equiv [a]$ of all real or (ordinary) complex numbers a, a general class $\mathfrak{P}$, and a binary relation R on the class $\mathfrak{P}$ which is transitive and has the composition property.

II. *As to Scope.*—The present study of the general limit (3) is arranged as follows:

§ 1. Elementary theorems.
§ 2. Necessary and sufficient conditions for the existence of a limit.
§ 3. Some modes of convergence.
§ 4. Quasi-limits. Upper and lower limits.
§ 5. Limits as to norm.
§ 6. Types of uniform convergence as to a general parameter.
§ 7. Double limits.
§ 8. Lemmas: Revised formulation of certain theorems of Fréchet.
§ 9. Composite range. Continuity.

III. *As to Notations.*—In order to expound briefly and luminously the considerable body of doctrine outlined in II, we make systematic use of readily understood notations for constantly recurring logical and mathematical notions.

For instance, the definition in I (3) of $\mathrm{L}\alpha = a$ we write: *there exists a system* $(p_e|e)$ *such that for every* $p\mathrm{R}p_e$ *it is true that* $|\alpha(p) - a| \leqq e$, or even more briefly, *there exists a system* $(p_e|e)$ *such that*

$$|\alpha(p) - a| \leqq e \qquad (p\mathrm{R}p_e).$$

Throughout e denotes a positive number.

§ 1. Elementary Theorems.

In the Introduction we have indicated the fundamental system

$$\Sigma = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R}^{\mathrm{on}\mathfrak{P}\mathfrak{P}.TC})$$

of notions under consideration, and defined the associated limit notion. We here add certain simple explanations and propositions.

It follows from R^C that for every element p there is an element p_p such that $p_p\mathrm{R}p$. For we have only to take in the definition of R^C $p_1 = p$, $p_2 = p$ and then take $p_p = p_3$.

In case $p\mathrm{R}p$ for every p, R is *reflexive*, in notation R^R. It is not assumed that R is reflexive. We shall however define an associated relation R_* which is transitive, has the composition property and is also reflexive. We define: $p_1\mathrm{R}_*p_2$ in case either $p_1\mathrm{R}p_2$ or $p_1 = p_2$. The proof that $\mathrm{R}_*{}^{TCR}$ is simple and is omitted.

To the definition of limit given in the introduction should be added the following one: α *converges to* $\sigma\infty$ ($\sigma = +$ or $-$) *as limit,* in notation:

$$\mathrm{L}_p\alpha(p) = \sigma\infty \qquad \text{or} \qquad \mathrm{L}\alpha = \sigma\infty,$$

or, with the relation R in evidence,

$$\mathrm{L}_{p\mathrm{R}}\alpha(p) = \sigma\infty \qquad \text{or} \qquad \mathrm{L}_{\mathrm{R}}\alpha = \sigma\infty$$

in case there exists a system $(p_e|e)$ *such that*

$$\sigma\alpha(p) \geqq e \qquad (p\mathrm{R}p_e).$$

The limit of α *exists* in case there exists a number a such that $\mathrm{L}\alpha = a$. The limit of α *exists finite or infinite* in case either $\mathrm{L}\alpha = +\infty$, $\mathrm{L}\alpha = -\infty$, or there is some a such that $\mathrm{L}\alpha = a$.

By $\alpha(\mathfrak{P}_0)$ will be denoted the function α *reduced to be on* $\mathfrak{P}_0 = [p_0]$, a subclass of $\mathfrak{P}$; that is, α considered only for elements p_0 of $\mathfrak{P}_0$. Every reduced function $\alpha(\mathfrak{P}_0)$ gives rise to a reduced limit $\mathrm{L}\alpha(\mathfrak{P}_0)$, if R as on $\mathfrak{P}_0\mathfrak{P}_0$ has the property C.* The definition of $\mathrm{L}\alpha(\mathfrak{P}_0)$ is the same as that of $\mathrm{L}\alpha$, except that all elements p must now be restricted to be of $\mathfrak{P}_0$.

By $\mathfrak{A}^*$ will be denoted $\mathfrak{A}$ enlarged by the addition of $+\infty$ and $-\infty$. Any element of $\mathfrak{A}^*$ will be denoted by a^*.

0. If $\mathrm{L}_{\mathrm{R}}\alpha = a^*$, then $\mathrm{L}_{\mathrm{R}_*}\alpha = a^*$, and conversely.

1. *If* $\mathrm{L}\alpha = a_1^*$ *and* $\mathrm{L}\alpha = a_2^*$, *then* $a_1^* = a_2^*$.

By proper choice of notation the possible cases may be reduced to six:

1) a_1^* and a_2^* both finite.

2σ) $a_1^* = \sigma\infty$, $\quad a_2^* = \sigma\infty$.

3σ) $a_1^* = \sigma\infty$, $\quad a_2^*$ finite.

4) $a_1^* = +\infty$, $\quad a_2^* = -\infty$.

Of these 3σ and 4 lead to contradictions, leaving only (1) to be considered since 2σ are in harmony with the theorem. Let us consider (1).

By hypothesis there are systems $(p_{1e}|e)$, $(p_{2e}|e)$ such that

$$|\alpha(p) - a_1^*| \leqq \frac{e}{2} \qquad (p\mathrm{R}p_{1e}),$$

$$|\alpha(p) - a_2^*| \leqq \frac{e}{2} \qquad (p\mathrm{R}p_{2e}).$$

By R^C there then exists a system $(p_e|e)$ such that $(p_e\mathrm{R}p_{1e},\ p_e\mathrm{R}p_{2e})$ for every e. Then

$$|a_1^* - a_2^*| \leqq |a_1^* - \alpha(p_e)| + |\alpha(p_e) - a_2^*| \leqq e$$

for every e, so that $|a_1^* - a_2^*| = 0$ and $a_1^* = a_2^*$.

2. *If* $\mathrm{L}\alpha = a$, *then* $\mathrm{L}|\alpha| = |a|$.†

3. *If* $\mathrm{L}\alpha = a$, *then* $\mathrm{L}(c\alpha) = ca \quad (c)$.

4. *If* $\mathrm{L}\alpha_1 = a_1$, $\mathrm{L}\alpha_2 = a_2$, *then*

$$\mathrm{L}(\alpha_1 + \alpha_2) = a_1 + a_2,$$

$$\mathrm{L}(\alpha_1\alpha_2) = a_1a_2,$$

$$\mathrm{L}\frac{\alpha_1}{\alpha_2} = \frac{a_1}{a_2} \qquad (a_2 \neq 0).$$

---

* The relation R as on $\mathfrak{P}_0\mathfrak{P}_0$ is necessarily T.

† Here $|\alpha|$, *the absolute of* α, denotes the function: $|\alpha|(p) = |\alpha(p)| \quad (p)$.

5. *If* $\mathrm{L}\alpha$ *exists and, for certain* cdp_0, $|\alpha(p) - c| \leqq d$ *for* $p\mathrm{R}p_0$, *then* $|\mathrm{L}\alpha - c| \leqq d$.

6. *If* $\alpha(p) \geqq c$ *for every* p *and* $\mathrm{L}\alpha$ *exists, then* $\mathrm{L}\alpha \geqq c$.

The proofs of propositions 2–6 are simple.

A single-valued function α is *monotone increasing* (*relative to* R) if $\alpha(p_1) \geqq \alpha(p_2)$ for every pair (p_1, p_2) such that $p_1\mathrm{R}p_2$; *properly* so if $\alpha(p_1) > \alpha(p_2)$ for every pair of distinct elements (p_1, p_2) such that $p_1\mathrm{R}p_2$. The term *monotone decreasing* is similarly defined.

7. *If* α *is single-valued and monotone, then* $\mathrm{L}\alpha$ *exists finite or infinite and is equal to* $\overline{B}\alpha^*$ *or* $\underline{B}\alpha$, *that is,* $\overline{B}_p\alpha(p)$ *or* $\underline{B}_p\alpha(p)$, *according as* α *is monotone increasing or monotone decreasing.*

Assume first that α is monotone increasing and that $\overline{B}_p\alpha(p) = a$, a finite number. Then there exists a system $(p_e|e)$ such that

$$a \geqq \alpha(p_e) \geqq a - e \qquad (e).$$

Then

$$a \geqq \alpha(p) \geqq \alpha(p_e) \geqq a - e \qquad (p\mathrm{R}p_e) \qquad (e);$$

and hence,

$$|\alpha(p) - a| \leqq e \qquad (p\mathrm{R}p_e) \qquad (e);$$

so that $\mathrm{L}\alpha = a$. The other cases are treated similarly.

§ 2. Necessary and Sufficient Conditions for the Existence of a Limit.

1. (Cauchy Condition). *In order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that there shall exist a system* $(p_e|e)$ *such that*

$$|\alpha(p_1) - \alpha(p_2)| \leqq e \qquad (p_1\mathrm{R}p_e,\ p_2\mathrm{R}p_e).$$

This condition is *necessary*. For there exists a system $(p_e|e)$ such that

$$|\mathrm{L}\alpha - \alpha(p)| \leqq \frac{e}{2} \qquad (p\mathrm{R}p_e) \qquad (e).$$

The condition is also *sufficient*. For there exists a sequence $\{p_n\}$ such that $p_{n+1}\mathrm{R}p_n$ $(n = 1, 2, 3, \cdots)$ and

$$|\alpha(p') - \alpha(p'')| \leqq \frac{1}{2^n}$$

for $(p'\mathrm{R}p_n,\ p''\mathrm{R}p_n)$. The numbers $\alpha(p_1)$, $\alpha(p_2)$, $\cdots$ form a limited or bounded set since

$$|\alpha(p_n) - \alpha(p_2)| \leqq |\alpha(p_n) - \alpha(p_{n-1})| + \cdots + |\alpha(p_3) - \alpha(p_2)|$$

$$\leqq \frac{1}{2^{n-2}} + \cdots + \frac{1}{2^1} \leqq 1,$$

for every $n > 2$. Hence there exists a subsequence $\{p_{n_m}\}$ such that the

* Read: *the (least) upper bound of* α.

numerical sequence $\{\alpha(p_{n_m})|m\}$ approaches some number a as a limit.* That $L\alpha = a$ now follows from the inequality

$$|\alpha(p) - a| \leqq |\alpha(p) - \alpha(p_{n_m})| + |\alpha(p_{n_m}) - a|$$

$$\leqq \frac{1}{2^{n_m}} + |\alpha(p_{n_m}) - a|,$$

which holds for $p \mathrm{R} p_{n_m}$.

The above theorem is true if R is replaced by R_* in the final line, as is evident by § 1, 0.

2. *In order that* $L\alpha$ *shall exist it is necessary and sufficient that*

$$L_{p_1 p_2}[\alpha(p_1) - \alpha(p_2)] = 0,$$

that is, that there exist systems $(p_{1e}|e)$, $(p_{2e}|e)$ such that

$$|\alpha(p_1) - \alpha(p_2)| \leqq e \qquad (p_1 \mathrm{R} p_{1e},\ p_2 \mathrm{R} p_{2e}) \qquad (e).$$

This theorem follows at once from the preceding and the composition property of R.

3. *In order that* $L\alpha$ *shall exist it is necessary and sufficient that there exist a system* $(p_e|e)$ *such that*

$$|\alpha(p) - \alpha(p_e)| \leqq e \qquad (p \mathrm{R} p_e) \qquad (e).$$

The necessity of this condition follows from the R_* form of theorem 1 and its sufficiency also on taking the p_e of the required $(p_e|e)$ of that theorem as the given $p_{e/2}$ of the $(p_e|e)$ of the present theorem.

4. *In order that* $L\alpha$ *shall not exist it is necessary and sufficient that there exist* e_0 *and a system* $(p_{1p}, p_{2p}|p)$ *such that*

$$p_{1p} \mathrm{R} p, \ p_{2p} \mathrm{R} p, \ |\alpha(p_{1p}) - \alpha(p_{2p})| > e_0 \dagger \qquad (p).$$

5. *In order that* $L\alpha$ *shall not exist it is necessary and sufficient that there exist* e_0 *and a system* $(p_p|p)$ *such that*

$$p_p \mathrm{R} p, \ |\alpha(p) - \alpha(p_p)| > e_0 \dagger \qquad (p).$$

Theorems 4 and 5 follow from 1 and 3 respectively.

The following theorems involve sequences $\{p_n\}$ and a property: *monotone* (R), of sequences and two binary relations R_0, R on the class of sequences. A sequence $\{p_n\}$ is *monotone* (R) in case $p_{n+1} \mathrm{R} p_n$ (n). The notation $\{p'_n\} \mathrm{R}_0 \{p''_n\}$ means that $p'_n \mathrm{R} p''_n$ (n). The notation $\{p'_n\} \mathrm{R} \{p''_n\}$ means that there exists a system $(n_n|n)$ such that $p'_{n_n} \mathrm{R} p''_n$ (n). Plainly if the relation R_0 holds for two sequences so does the relation R.

* The ordinary properties of $L_{n \to \infty}$ are here assumed known.

† If α is multiply valued, this inequality is understood to hold for at least one mode of determining the functional values involved.

6. *If* $L\alpha = a^*$, *there exists a sequence* $\{p_n^0\}$ *monotone* (R) *such that* $L_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}R_0\{p_n^0\}$, *and also for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}R\{p_n^0\}$.

Take $\{p_n^0\}$ such that $p_{n+1}^0 R p_n^0$ (n) and

$$|a^* - \alpha(p)| \leqq \frac{1}{n} \qquad (pRp_n^0) \qquad (n)$$

or

$$\sigma\alpha(p) \geqq n \qquad (pRp_n^0) \qquad (n),$$

according as a^* is finite or equals $\sigma\infty$, $\sigma = \pm$.

7. $L\alpha = a^*$ *if there exists a sequence* $\{p_n^0\}$ *such that* $L_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}R_0\{p_n^0\}$.

We prove the equivalent contrapositive theorem:

7′. *If it is untrue that* $L\alpha = a^*$, *then for every sequence* $\{p_n\}$ *there exists a sequence* $\{p_n^0\}$ *monotone* (R) *such that* $\{p_n^0\}R_0\{p_n\}$ *and it is untrue that* $L_n\alpha(p_n^0) = a^*$.

We have given e_0 and $(p_p|p)$ such that $p_p R p$ (p) and (for at least one determination of $\alpha(p_p)$)

$$|a^* - \alpha(p_p)| > e_0 \qquad (p)$$

or

$$\sigma\alpha(p_p) < e_0 \qquad (p)$$

according as a^* is finite or equals $\sigma\infty$, $\sigma = \pm$. Hence from the given sequence $\{p_n\}$ an effective sequence $\{p_n^0\}$ is obtained by recursion in the form $\{p_{p'_n}\}$ on taking $p'_1 = p_1$ and for $n > 1$ $p'_n R(p_n, p_{n-1}^0)$.

8. *The following conditions on* a^* *and the function* α *are equivalent*:

(*A*) $L\alpha = a^*$;

(*B*) *There exists a sequence* $\{p_n^0\}$ *such that* $L_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}R_0\{p_n^0\}$;

(*C*) *There exists a sequence* $\{p_n^0\}$ *such that* $L_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}R_0\{p_n^0\}$;

(*D*) *There exists a sequence* $\{p_n^0\}$ *such that* $L_n\alpha(p_n) = a^*$ *for every sequence* $\{p_n\}$ *monotone* (R) *such that* $\{p_n\}R\{p_n^0\}$.

This follows from 6 and 7. For by 6 *A* implies *B* and *D*; *B* implies *C*; *D* implies *C*; and by 7 *C* implies *A*.

9. *In order that* $L\alpha$ *shall exist it is necessary and sufficient that there exist a sequence* $\{p_n^0\}$ *such that* $L_n[\alpha(p_n^0) - \alpha(p_n)] = 0$ *for every sequence* $\{p_n\}$ *such that* $\{p_n\}R_0\{p_n^0\}$.

The necessity follows from 8. The sufficiency is equivalent to

9′. *If* $L\alpha$ *does not exist, then for every sequence* $\{p_n\}$ *there exists a sequence* $\{p_n^0\}$ *such that* $\{p_n^0\}R_0\{p_n\}$ *and it is untrue that* $L_n[\alpha(p_n) - \alpha(p_n^0)] = 0$.

In proof of 9′ we have given a sequence $\{p_n\}$ and by 5 a number e_0 and a system $(p_p|p)$ such that for every p $p_p \mathrm{R} p$ and $|\alpha(p) - \alpha(p_p)| > e_0$.* Then, taking the sequence $\{p_{p_n}\}$ as sequence $\{p_n^0\}$, we have for every n $p_n^0 \mathrm{R} p_n$ and $|\alpha(p_n) - \alpha(p_n^0)| > e_0$,† and accordingly, as stated, $\{p_n^0\} \mathrm{R}_0 \{p_n\}$ and the untruth of $\mathrm{L}_n[\alpha(p_n) - \alpha(p_n^0)] = 0$.

§ 3. Some Modes of Convergence.

The non-existence of $\mathrm{L}\alpha$ implies by § 2, 5 the existence of a sequence‡ $\{p_n\}$ monotone (R) such that $\mathrm{L}_n\alpha(p_n)$ does not converge, and for every sequence $\{p_n^0\}$ the existence of a sequence§ $\{p_n\}$ monotone (R) such that $\{p_n\} \mathrm{R} \{p_n^0\}$ and $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| = \infty$. Hence we are led to formulate the following *modes of existence* (or *convergence*) of $\mathrm{L}\alpha$.

$\mathrm{L}\alpha$ *exists absolutely* in case there exists a sequence $\{p_n^0\}$ such that $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| < \infty$ for every sequence $\{p_n\}$ monotone (R) such that $\{p_n\} \mathrm{R} \{p_n^0\}$.

$\mathrm{L}\alpha$ *exists unconditionally* in case $\mathrm{L}_n\alpha(p_n)$ converges finitely (or what is equivalent, in case $\Sigma_n[\alpha(p_{n+1}) - \alpha(p_n)]$ converges finitely) for every sequence $\{p_n\}$ monotone (R).

$\mathrm{L}\alpha$ *exists absolutely-unconditionally* in case $\Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)| < \infty$ for every sequence $\{p_n\}$ monotone (R).

$|\mathrm{L}|\alpha = a$, *the absolute limit of* α *is* a, in case a is the least upper bound of $|\alpha(p_1)| + \Sigma_n|\alpha(p_{n+1}) - \alpha(p_n)|$ for all sequences $\{p_n\}$ monotone (R). Here it would be simpler and equivalent to consider finite (instead of infinite) monotone sequences $\{p_n\}$.

By the initial remark, $\mathrm{L}\alpha$ exists if $\mathrm{L}\alpha$ exists absolutely or unconditionally. Evidently $\mathrm{L}\alpha$ exists absolutely and unconditionally if $\mathrm{L}\alpha$ exists absolutely-unconditionally. Finally, if $|\mathrm{L}|\alpha$ exists, then $\mathrm{L}\alpha$ exists absolutely-unconditionally and clearly $|\mathrm{L}|\alpha \geqq |\mathrm{L}\alpha|$.

That $\mathrm{L}\alpha$ *exist unconditionally it is necessary and sufficient that* $\mathrm{L}\alpha(\mathfrak{P}_0)$ *exist for every subclass* $\mathfrak{P}_0$ *of* $\mathfrak{P}$ *such that* R *as on* $\mathfrak{P}_0\mathfrak{P}_0$ *has the property C.* The condition is necessary, since $\mathrm{L}\alpha$, and *a fortiori* $\mathrm{L}\alpha(\mathfrak{P}_0)$, exists unconditionally, and therefore $\mathrm{L}\alpha(\mathfrak{P}_0)$ exists. To prove it sufficient, that is, that $\mathrm{L}_n\alpha(p_n)$ exists for every $\{p_n\}$ monotone (R), take $\mathfrak{P}_0 = [p_n|n]$. Then there exists a system $(n_e|e)$ such that $|\mathrm{L}\alpha(\mathfrak{P}_0) - \alpha(p_n)| \leqq e$ for every $p_n \mathrm{R} p_{n_e}$, in particular, for every p_n such that $n \geqq n_e + 1$. Hence $\mathrm{L}_n\alpha(p_n)$ exists.

It is readily seen from the second paragraph of § 3 that if $\mathrm{L}\alpha$ exists then as to the existence of $\mathrm{L}\alpha$ absolutely, $\mathrm{L}\alpha$ unconditionally, $\mathrm{L}\alpha$ absolutely-

* For at least one determination of $\alpha(p)$ and $\alpha(p_p)$.

† For at least one determination of $\alpha(p_n)$, $\alpha(p_n^0)$.

‡ Take p_1 at random and for every n $p_{n+1} = p_{p_n}$.

§ Take p_1 at random, $p_2 \mathrm{R}(p_1, p_1^0, p_2^0)$, and for every n $p_{2n+1} = p_p$, $p_{2n+2} \mathrm{R}(p_{2n+1}, p_{2n+1}^0, p_{2n+2}^0)$.

unconditionally, $|\mathrm{L}|\alpha$, only the six situations can occur which are indicated in the table below.

	$\mathrm{L}\alpha$	$\mathrm{L}\alpha$ abs.	$\mathrm{L}\alpha$ unc.	$\mathrm{L}\alpha$ abs.-unc.	$\|\mathrm{L}\|\alpha$
(I)	+	+	+	+	+
(II)	+	+	+	+	−
(III)	+	+	+	−	−
(IV)	+	+	−	−	−
(V)	+	−	+	−	−
(VI)	+	−	−	−	−

Here a + sign indicates the existence of the concept at the head of the column, a − sign its non-existence. That these six situations actually occur is shown by the following examples (I) ⋯ (VI).

Let $A = a_1 + a_2 + \cdots = \mathrm{L}_n A_n$ be an absolutely convergent series. Let $C = c_1 + c_2 + \cdots = \mathrm{L}_n C_n$ be a conditionally convergent series. Denote by $\mathfrak{P}^{\mathrm{III}}$ the class $[n]$ of positive integers n.

(I) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case $n_1 \geqq n_2$. $\alpha(n) = A - A_n$ (n).

(II) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}} + \infty$. $n_1 \mathrm{R} n_2$ in case $n_1 = n_2$; $\infty \mathrm{R} \infty$; $\infty \mathrm{R} n$ (n). $\alpha(n) = n$ (n); $\alpha(\infty) = 0$.

(III) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case either n_1 odd n_2 even or $n_1 \equiv n_2 \pmod 2$ with $n_1 \geqq n_2$. $\alpha(n) = C - C_{(n+1)/2}$ (n odd), $A - A_{n/2}$ (n even).

(IV) The same as (III) except that $\alpha(n) = n$ (n odd), $A - A_{n/2}$ (n even).

(V) $\mathfrak{P} = \mathfrak{P}^{\mathrm{III}}$. $n_1 \mathrm{R} n_2$ in case $n_1 \geqq n_2$. $\alpha(n) = C - C_n$ (n).

(VI) The same as (III) except that $\alpha(n) = n$ (n odd), $C - C_{n/2}$ (n even)

§ 4. Quasi-Limits. Upper and Lower Limits.

***L*$\alpha = a^*$, *a quasi-limit of* α *is* a^***, in case there exists a system $(p_{ep}|ep)$ such that $p_{ep} \mathrm{R} p$ (ep) and

$$|a^* - \alpha(p_{ep})| \leqq e \quad (ep) \qquad \text{or} \qquad \sigma\alpha(p_{ep}) \geqq e \quad (ep),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

***L*$_0\alpha = a^*$, *a weak quasi-limit of* α *is* a^***, in case there exists a system $(p_e|e)$ such that

$$|a^* - \alpha(p_e)| \leqq e \quad (e) \qquad \text{or} \qquad \sigma\alpha(p_e) \geqq e \quad (e),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

If $\mathfrak{P}_0$ is a subclass of $\mathfrak{P}$ such that $\mathfrak{P}$ as on $\mathfrak{P}_0\mathfrak{P}_0$ has the properties *TC*, the notations ***L*$\alpha(\mathfrak{P}_0)$, *L*$_0\alpha(\mathfrak{P}_0)$** have obvious meanings. For two subclasses $\mathfrak{P}_1\mathfrak{P}_2$ of $\mathfrak{P}$ denote by $\mathfrak{P}_1 \mathrm{R} \mathfrak{P}_2$ the condition that for every p_2 of $\mathfrak{P}_2$ there is a p_1 of $\mathfrak{P}_1$ for which $p_1 \mathrm{R} p_2$. Hence if $\mathfrak{P}_0 \mathrm{R} \mathfrak{P}$ R as on $\mathfrak{P}_0\mathfrak{P}_0$ has the properties *TC*.

1. *If* ***L*$\alpha = a^*$, *then* *L*$_0\alpha = a^*$.**

2. *In order that* $\mathrm{L}\alpha = a^*$, *it is necessary and sufficient* that ***L*$\alpha(\mathfrak{P}_0) = a^*$ *uniquely for every* $\mathfrak{P}_0 \mathrm{R} \mathfrak{P}$.**

The condition is *necessary.* Since $\mathrm{L}\alpha = a^*$ and $\mathfrak{P}_0\mathrm{R}\mathfrak{P}$ there is a system $(p_e|e)_0$† such that

$$|a^* - \alpha(p)| \leqq e \quad (p\mathrm{R}p_e) \qquad \text{or} \qquad \sigma\alpha(p) \geqq e \quad (p\mathrm{R}p_e),$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$. Then by means of a system $(p_{ep}|ep)_0$ such that $p_{ep}\mathrm{R}(p, p_e)$ $(ep)_0$, it is clear that $\boldsymbol{L}\alpha(\mathfrak{P}_0) = a^*$. It remains to show that $\boldsymbol{L}\alpha(\mathfrak{P}_0) = a_1^*$ implies $a_1^* = a^*$. From this hypothesis there is a system $(p_{1ep}|ep)_0$ such that we have for every e

$$|a_1^* - \alpha(p_{1ep_e})| \leqq \frac{e}{3} \qquad \text{or} \qquad \sigma_1\alpha(p_{1ep_e}) \geqq \frac{e}{3};$$

$$|a^* - \alpha(p_{1ep_e})| \leqq e \qquad \text{or} \qquad \sigma\alpha(p_{1ep_e}) \geqq e$$

(according to the values of a_1^*; a^*), whence the conclusion $a_1^* = a^*$ follows readily.

The condition is *sufficient.* The proof is indirect. The untruth of $\mathrm{L}\alpha = a^*$ implies the existence of a positive number e_0 and a system $(p_p|p)$ such that

$$p_p\mathrm{R}p \quad (p), \qquad |\alpha(p_p) - a^*| > e_0 \quad (p) \qquad \text{or} \qquad \sigma\alpha(p_p) < e_0 \quad (p),$$

according to the value of a^*. Since $\boldsymbol{L}\alpha = a^*$, $\mathfrak{P}$ being a $\mathfrak{P}_0\mathrm{R}\mathfrak{P}$, there is a system $(p_{ep}|ep)$ such that

$$p_{ep}\mathrm{R}p \quad (ep), \qquad |\alpha(p_{ep}) - a^*| \leqq e \quad (ep) \qquad \text{or} \qquad \sigma\alpha(p_{ep}) \geqq e \quad (ep),$$

according to the value of a^*. The class

$$\mathfrak{P}_0 \equiv [p_0] \equiv [p'_{ep} \equiv p_{p_{ep}}|ep]$$

is a class $\mathfrak{P}_0\mathrm{R}\mathfrak{P}$. Hence by hypothesis $\boldsymbol{L}\alpha(\mathfrak{P}_0) = a^*$, whereas evidently

$$|\alpha(\mathfrak{P}_0) - a^*| > e_0 \qquad \text{or} \qquad \sigma\alpha(\mathfrak{P}_0) < e_0.$$

This is the desired contradiction. Hence $\mathrm{L}\alpha = a^*$, as stated.

We assume α to be real- and single-valued in the remainder of § 4.

Associated with α are two functions $\overline{\alpha}$; $\underline{\alpha}$ (read: α *upper*; α *lower*) on $\mathfrak{P}$ to $\mathfrak{A}^*$, with

$$\overline{\alpha}(p) \equiv \overline{B}_{p_1|p_1\mathrm{R}p}\,\alpha(p_1); \qquad \underline{\alpha}(p) \equiv \underline{B}_{p_1|p_1\mathrm{R}p}\,\alpha(p_1),$$

that is, $\overline{\alpha}(p)$ is the (least) upper bound of $\alpha(p_1)$ for all $p_1\mathrm{R}p$; $\underline{\alpha}(p)$ is the (greatest) lower bound of $\alpha(p_1)$ for all $p_1\mathrm{R}p$. Plainly $\overline{\alpha}$; $\underline{\alpha}$ are monotone functions decreasing; increasing, and for every p, $\underline{\alpha}(p) \leqq \overline{\alpha}(p)$.

The lower bound $\underline{B}\overline{\alpha} \equiv \underline{B}_p\overline{\alpha}(p)$ of the function $\overline{\alpha}$ is *the upper limit* $\overline{\mathrm{L}}\alpha$ of the function α. Similarly the upper bound $\overline{B}\underline{\alpha}$ of $\underline{\alpha}$ is *the lower limit* $\underline{\mathrm{L}}\alpha$ of α. It is readily seen that $\underline{\mathrm{L}}\alpha \leqq \overline{\mathrm{L}}\alpha$.

3. $\mathrm{L}\overline{\alpha} = \overline{\mathrm{L}}\alpha$; $\mathrm{L}\underline{\alpha} = \underline{\mathrm{L}}\alpha$.

† In this proof the suffix 0 indicates that the elements p_e, p, p_{ep}, etc., involved belong to $\mathfrak{P}_0$.

This follows from § 1, 7.

4_1. *In order that* $\overline{\mathrm{L}}\alpha = a$, *it is necessary and sufficient that there exist systems* $(p_e|e)$, $(p_{ep}|ep)$ *such that*

(1) $$\alpha(p) \leqq a + e \quad (p\mathrm{R}p_e) \quad (e),$$

(2) $$p_{ep}\mathrm{R}p, \qquad \alpha(p_{ep}) \geqq a - e \quad (ep).$$

It is *necessary*. For there are systems $(p_e|e)$, $(p_{ep}|ep)$ such that $a \leqq \overline{\alpha}(p_e) \leqq a + e$ (e), $p_{ep}\mathrm{R}p$ (ep), $\overline{\alpha}(p) - \alpha(p_{ep}) \leqq e$ (ep). These systems satisfy (1), (2). For $\alpha(p) \leqq \overline{\alpha}(p_e) \leqq a + e$ $(p\mathrm{R}p_e)$ (e), and $a \leqq \overline{\alpha}(p) \leqq \alpha(p_{ep}) + e$ (ep).

It is *sufficient*. For by (1) $\overline{\alpha}(p_e) \leqq a + e$ (e); hence $\overline{\mathrm{L}}\alpha \leqq (\alpha p_e) \leqq a + e$ (e), and accordingly $\overline{\mathrm{L}}\alpha \leqq a$; and by (2) $\overline{\alpha}(p) \geqq \alpha(p_{ep}) \geqq a - e$ (ep); hence $\overline{\alpha}(p) \geqq a$ (p), and accordingly $\overline{\mathrm{L}}\alpha \geqq a$. Hence $\overline{\mathrm{L}}\alpha = a$, as stated.

4_2. *In order that* $\underline{\mathrm{L}}\alpha = a$, *it is necessary and sufficient that there exist systems* $(p_e|e)$, $(p_{ep}|ep)$ *such that*

(1) $$\alpha(p) \geqq a - e \quad (p\mathrm{R}p_e) \quad (e),$$

(2) $$p_{ep}\mathrm{R}p, \qquad \alpha(p_{ep}) \leqq a + e \quad (ep).$$

5. $\mathrm{L}\alpha = \alpha^*$ *is equivalent to* $\overline{\mathrm{L}}\alpha = \underline{\mathrm{L}}\alpha = a^*$.

6. $\boldsymbol{L}\alpha = \overline{\mathrm{L}}\alpha$, $\boldsymbol{L}\alpha = \underline{\mathrm{L}}\alpha$.

7. $\underline{\mathrm{L}}\alpha \leqq \boldsymbol{L}\alpha \leqq \overline{\mathrm{L}}\alpha$ *for every* $\boldsymbol{L}\alpha$ *finite or infinite.*

Theorems 5, 6, 7 follow readily* from the definitions and theorems 3, 4.

8. $\mathrm{L}\alpha = a^*$ *is equivalent to* $\boldsymbol{L}\alpha = a^*$ *uniquely.*

This theorem follows from 5, 6, 7.

§ 5. Limits as to Norm.

As basis for § 5 we take the system

$$\Sigma_1 = (\mathfrak{A};\ \mathfrak{P};\ \mathrm{R};\ \nu),$$

where $\mathfrak{A}$, $\mathfrak{P}$, R have the same meanings as before and are subject to the same postulates, and where ν, *the norm,* is a real-valued function on $\mathfrak{P}$ to $\mathfrak{A}$ which is monotone decreasing (R) and such that $\underline{B}\nu = 0$.

$\mathrm{L}_\nu\alpha = a^*$, *the limit of* α *is* a^* *as to the norm* ν, provided there exists a system $(d_e|e)$ such that for every p such that $\nu(p) \leqq d_e$

$$|a^* - \alpha(p)| \leqq e \qquad \text{or} \qquad \sigma\alpha(p) \geqq e,$$

according as a^* is finite or is $\sigma\infty$, $\sigma = \pm$.

Associated with ν is a relation R′ (having the properties T, C of the relation R) defined as follows: $p_1\mathrm{R}'p_2$ if and only if $\nu(p_1) \leqq \nu(p_2)$. It is

* E.g., if $\underline{\mathrm{L}}\alpha = a$, from the systems $(pe|e)(p_{ep}|_e p)$ of 4_2 we obtain a system $(p^\circ_{ep}|ep)$ effective for the proof of 6_2: $\boldsymbol{L}\alpha = \underline{\mathrm{L}}\alpha = a$, as follows: For every ep take $p'_{ep}\mathrm{R}(p,\ p_e)$ and $p^\circ_{ep} \equiv p_{ep}{}'_{ep}$.

easily shown that L_ν and $L_{R'}$ are equivalent: Whenever either $L_\nu\alpha$ or $L_{R'}\alpha$ exists finite or infinite, the other does and the two limits are equal. Hence the preceding theorems may be applied to L_ν. In particular L_ν is unique.

We are concerned in the sequel with certain relations between L_ν and L_R.

1. *If* $L_\nu\alpha = a^*$, *then* $L_R\alpha = a^*$.

Assume a^* finite. Then there exist systems $(d_e|e)$, $(p_e|e)$ such that

$$|a^* - \alpha(p)| \leqq e \qquad (p \text{ with } \nu(p) \leqq d_e) \quad (e)$$

and $\nu(p_e) \leqq d_e$ (e). Then $|a^* - \alpha(p)| \leqq e$ (pRp_e), as stated, since $\nu(p) \leqq \nu(p_e) \leqq d_e$ (pRp_e). The proof is similar for the case a^* infinite.

2. *If* $L_R\alpha$ *exists, then in order that* $L_\nu\alpha$ *shall exist and equal* $L_R\alpha$, *it is necessary and sufficient that there exist systems* $(d_e|e)$, $(p_{ep_1p}|ep_1p)$ *such that*

$$p_{ep_1p}Rp_1, \quad |\alpha(p_{ep_1p}) - \alpha(p)| \leqq e \quad (e, p_1, p \text{ with } \nu(p) \leqq d_e).$$

The condition is *necessary.* Take systems $(d_e|e)$, $(p_e|e)$, $(p_{ep_1p}|ep_1p)$ such that

$$|\alpha(p) - L_\nu\alpha| \leqq \frac{e}{2} \quad (p \text{ with } \nu(p) \leqq d_e) \quad (e),$$

$$\nu(p_e) \leqq d_e \quad (e),$$

$$p_{ep_1p}Rp_1, \qquad p_{ep_1p}Rp_e \quad (ep_1p).$$

(The p_{ep_1p} may clearly be taken as the same for every p.) Then for every p_1 and e and every p with $\nu(p) \leqq d_e$, we have

$$p_{ep_1p}Rp_1, \quad |\alpha(p) - \alpha(p_{ep_1p})| \leqq |\alpha(p) - L_\nu\alpha| + |L_\nu\alpha - \alpha(p_{ep_1p})| \leqq e,$$

as stated.

The condition is *sufficient.* Take $(p_e|e)$ such that $|\alpha(p) - L_R\alpha| \leqq e/2$ (pRp_e). Then for every e and p with $\nu(p) \leqq d_{e/2}$, we have

$$|\alpha(p) - L_R\alpha| \leqq |\alpha(p) - \alpha(p_{(e/2)p_ep})| + |\alpha(p_{(e/2)p_ep}) - L_R\alpha| \leqq e,$$

as stated.

All of the classical limits are limits as to a norm, and as such are instances of our general limit. We consider, as examples, the Riemann and Lebesgue integrals.

Let f denote a function of the variable x, x ranging over I: $a \leqq x \leqq b$. Take $\mathfrak{P}$ to be the class of all partitions p of I; a partition of I is a set $I_1, \cdots, I_n$ of intervals non-overlapping (except for end points) such that $I = I_1 + \cdots + I_n$. We define $\nu(p)$ for a partition $p = I_1, \cdots, I_n$ as the length of the longest I_k $(k = 1, \cdots, n)$. We say a partition $p' = I_1', \cdots, I_{n'}'$ is in the R relation to a partition $p'' = I_1'', \cdots, I_{n''}''$ if p' is a repartition of p'', that is, if every I_k' lies entirely in some I_k'' (except possibly for end points). Every function f gives rise to an associated (multiply

valued) function α on $\mathfrak{P}$ to $\mathfrak{A}$: $\alpha(p) \equiv \alpha(I_1, \cdots, I_n) \equiv \sum_k^{1n} f(x_k) I_k$, where I_k denotes the length (or measure) of I_k and x_k is any point of I_k. The ordinary Riemann definition of integration for a bounded function f is, except for form, as follows: If the function f is such that for the associated function α, $L_\nu\alpha$ exists, then $L_\nu\alpha$ is called the (Riemann) integral of f from a to b and is denoted by $\int_a^b f(x)dx$. But it follows from 1 and 2 above that $L_R\alpha$ used in place of $L_\nu\alpha$ in the definition just given would yield a second and equivalent definition.

This remark is important in that it leads to a simple and natural definition of the Lebesgue integral $\int_a^b f(x)dx$ as a limit. To secure such a definition from the second definition it is only necessary to define partition differently, a partition now being a set of non-overlapping measurable sets $I_1, \cdots, I_n$ such that $I = I_1 + \cdots + I_n$. The junior author hopes soon to publish a theory of integration from this point of view which has been in his possession since 1917. acc TAMS V. 27 (1925) p. 491-515.

§ 6. Types of Uniform Convergence as to a General Parameter.

The *fundamental system* for § 6 is

$$\Sigma_2 \equiv (\mathfrak{A};\ \mathfrak{P};\ R^{\text{on } \mathfrak{P}, TC};\ \mathfrak{Q}),$$

that is, the fundamental system of § 1 with the adjunction of a *general class* $\mathfrak{Q} \equiv [q]$ of elements q.

We consider functions $\alpha \equiv (\alpha(p)\,|\,p)$ on $\mathfrak{P}$ to $\mathfrak{A}$; $\beta \equiv (\beta(q)\,|\,q)$ on $\mathfrak{Q}$ to $\mathfrak{A}$; $\varphi \equiv (\varphi(pq)\,|\,pq)$ on $\mathfrak{P}\mathfrak{Q}$ to $\mathfrak{A}$. Thus a function φ is a function of the variables pq which range independently over the (conceptually, but not necessarily actually, distinct) classes $\mathfrak{P}\mathfrak{Q}$; we denote by $\varphi(\diamond q) \equiv (\varphi(pq)\,|\,p)$, $\varphi(p\diamond) \equiv (\varphi(pq)\,|\,q)$ the functions α on $\mathfrak{P}$ to $\mathfrak{A}$, β on $\mathfrak{Q}$ to $\mathfrak{A}$ obtained from φ by fixing the respective arguments q, p. With respect to the limits now to be defined the argument q plays the rôle of a parameter.

$L\varphi = \beta$ ($\mathfrak{Q}$; unif.), *the limit of φ is β over $\mathfrak{Q}$ uniformly,* in case there exists a system $(p_e\,|\,e)$ such that

$$|\varphi(p\diamond) - \beta| \leqq e \quad (pRp_e) \quad (e).$$

$L\varphi = \beta$ ($\mathfrak{Q}$; quasi-unif.), *the limit of φ is β over $\mathfrak{Q}$ quasi-uniformly,* in case there exists a system $(p_{eq}\,|\,eq)$ such that

(1) the set $[p_{eq}\,|\,q]$ is finite (e),
(2) $|\varphi(pq) - \beta(q)| \leqq e \quad (pRp_{eq}) \quad (eq)$.

$L\varphi = \beta$ ($\mathfrak{Q}$; semi-unif.), *the limit of φ is β over $\mathfrak{Q}$ semi-uniformly,* in case there exists a system $(p_{eq}\,|\,eq)$ such that

(1) the set $[p_{eq}\,|\,q]$ is (finitely or infinitely) denumerable (e),
(2) $|\varphi(pq) - \beta(q)| \leqq e \quad (pRp_{eq}) \quad (eq)$.

$\boldsymbol{L}\varphi = \beta$ ($\mathfrak{Q}$; unif.), *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *uniformly*, in case there exists a system $(p_{ep}|ep)$ such that

$$p_{ep}\mathrm{R}p, \quad |\varphi(p_{ep}\diamond) - \beta| \leqq e \quad (ep).$$

$\boldsymbol{L}\varphi = \beta$ ($\mathfrak{Q}$; quasi-unif.), *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *quasi-uniformly*, in case there exists a system $(p_{epq}|epq)$ such that

(1) the set $[p_{epq}|q]$ is finite (ep),
(2) $p_{epq}\mathrm{R}p, \quad |\varphi(p_{epq}q) - \beta(q)| \leqq e \quad (epq)$.

$\boldsymbol{L}\varphi = \beta$ ($\mathfrak{Q}$; semi-unif.), *a quasi-limit of* φ *is* β *over* $\mathfrak{Q}$ *semi-uniformly*, in case there exists a system $(p_{epq}|epq)$ such that

(1) the set $[p_{epq}|q]$ is denumerable (ep),
(2) $p_{epq}\mathrm{R}p, \quad |\varphi(p_{epq}q) - \beta(q)| \leqq e \quad (epq)$.

From the definitions of the three quasi-limits $\boldsymbol{L}$ we obtain definitions of the corresponding *weak quasi-limits* $\boldsymbol{L}_0$ by omitting the subscript p and the conditions involving R.

These types of uniform convergence as to a parameter are for use in §§ 7, 9. Obviously the quasi-uniform convergence L and the uniform convergence L are equivalent, and the quasi-uniform convergences L, $\boldsymbol{L}$, $\boldsymbol{L}_0$ imply the semi-uniform convergences L, $\boldsymbol{L}$, $\boldsymbol{L}_0$ respectively.

§ 7. Double Limits.

From two systems

$\Sigma' \equiv (\mathfrak{A};\ \mathfrak{P}' \equiv [p'];\ \mathrm{R}'^{\text{on } \mathfrak{P}'\mathfrak{P}'.TC})$, $\Sigma'' \equiv (\mathfrak{A};\ \mathfrak{P}'' \equiv [p''];\ \mathrm{R}''^{\text{on } \mathfrak{P}''\mathfrak{P}''.TC})$

of the type studied in §§ 1–4 we form the *composite system*

$$\Sigma \equiv (\mathfrak{A};\ \mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}'';\ \mathrm{R} \equiv \mathrm{R}'\mathrm{R}''^{\text{on } \mathfrak{P}\mathfrak{P}.TC})$$

of the same type. Here $\mathfrak{P}'$ $\mathfrak{P}''$ are two general classes conceptually (but not necessarily actually) distinct; $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ is the product or composite class $[p] \equiv [p'p'']$ of all composite elements $p \equiv p'p''$; and $\mathrm{R} \equiv \mathrm{R}'\mathrm{R}''$ is the composite relation R on $\mathfrak{P}\mathfrak{P}$: $p_1\mathrm{R}p_2 \equiv p_1'p_1''\mathrm{R}p_2'p_2''$ in case $p_1'\mathrm{R}'p_2'$ and $p_1''\mathrm{R}''p_2''$.

We denote limits, quasi-limits, weak quasi-limits as to R′; R″; R of functions α'; α''; α on $\mathfrak{P}'$; $\mathfrak{P}''$; $\mathfrak{P}$ to $\mathfrak{A}$ by L′, $\boldsymbol{L}'$, $\boldsymbol{L}_0'$; L″, $\boldsymbol{L}''$, $\boldsymbol{L}_0''$; L, $\boldsymbol{L}$, $\boldsymbol{L}_0$, respectively. For a function α on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ L′α is said to exist in case for every p'' L′$\alpha(\diamond p'')$ exists, and in this case L′α denotes the function $(\mathrm{L}'\alpha(\diamond p'')|p'')$ on $\mathfrak{P}''$ to $\mathfrak{A}$; etc.

The *iterated double limits* L′L″, L″L′ and *quasi-limits* $\boldsymbol{L}'$L″, $\boldsymbol{L}_0'$L″, $\boldsymbol{L}''$L′, $\boldsymbol{L}_0''$L′ have evident definitions. The *simultaneous double limit* (L′L″) ≡ (L″L′) and *quasi-limits* $(\boldsymbol{L}'\boldsymbol{L}'') \equiv (\boldsymbol{L}''\boldsymbol{L}')$, $(\boldsymbol{L}_0'\boldsymbol{L}_0'') \equiv (\boldsymbol{L}_0''\boldsymbol{L}_0')$ are defined as the simple limit L and quasi-limits $\boldsymbol{L}$, $\boldsymbol{L}_0$ for the composite

system Σ. The *simultaneous double limits* $(\boldsymbol{L}'\mathrm{L}'') \equiv (\mathrm{L}''\boldsymbol{L}')$, $(\boldsymbol{L}_0'\mathrm{L}'') \equiv (\mathrm{L}''\boldsymbol{L}_0')$ are defined as follows:

$(\boldsymbol{L}'\mathrm{L}'')\alpha = a$, in case there exists a system $(p'_{ep'}, p''_{ep'} | ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \alpha(p'_{ep'}p'')| \leqq e \quad (p''\mathrm{R}''p''_{ep'})(ep');$$

$(\boldsymbol{L}_0'\mathrm{L}'')\alpha = a$, in case there exists a system $(p'_e, p''_e | e)$ such that

$$|a - \alpha(p'_e p'')| \leqq e \quad (p''\mathrm{R}''p''_e) \quad (e).$$

The limits $(\boldsymbol{L}''\mathrm{L}')$, $(\boldsymbol{L}_0''\mathrm{L}')$ are defined similarly.

1. *If* $\mathrm{L}\alpha$ *and* $\mathrm{L}'\alpha$ *exist, then* $\mathrm{L}''\mathrm{L}'\alpha$ *exists and is equal to* $\mathrm{L}\alpha$.

2. *If* $\mathrm{L}'\alpha$ *and* $\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}\alpha$ *shall exist it is necessary and sufficient that* $\mathrm{L}(\alpha - \mathrm{L}'\alpha) = 0$.

The condition is *necessary.** For by 1 $\mathrm{L}''\mathrm{L}'\alpha$ exists, equal to $\mathrm{L}\alpha$. Hence $\mathrm{L}\mathrm{L}'\alpha$ exists, equal to $\mathrm{L}\alpha$, and

$$\mathrm{L}(\alpha - \mathrm{L}'\alpha) = \mathrm{L}\alpha - \mathrm{L}\mathrm{L}'\alpha = \mathrm{L}\alpha - \mathrm{L}\alpha = 0.$$

The condition is *sufficient.* From the hypotheses we readily secure a system $(p'_e, p''_{1e}, p''_{2e}, p''_e | e)$ such that for every e

$$|\alpha(p'p'') - \mathrm{L}'\alpha(\diamond p'')| \leqq \frac{e}{3} \quad (p'\mathrm{R}'_* p'_e,\ p''\mathrm{R}''_* p''_{1e}),$$

$$|\alpha(p'_e p''_1) - \alpha(p'_e p''_2)| \leqq \frac{e}{3} \quad (p''_1\mathrm{R}''p''_{2e},\ p''_2\mathrm{R}''p''_{2e}),$$

$$p''_e\mathrm{R}''p''_{1e}, \quad p''_e\mathrm{R}''p''_{2e}.$$

Then for every e $(p'\mathrm{R}'p'_e,\ p''\mathrm{R}''p''_e)$ implies

$$\begin{aligned}|\alpha(p'p'') - \alpha(p'_e p''_e)| &\leqq |\alpha(p'p'') - \mathrm{L}'\alpha(\diamond p'')| \\ &\quad + |\mathrm{L}'\alpha(\diamond p'') - \alpha(p'_e p'')| + |\alpha(p'_e p'') - \alpha(p'_e p''_e)| \leqq e,\end{aligned}$$

and hence by § 2, 3 $\mathrm{L}\alpha$ exists, as stated.

3. *If* $\mathrm{L}'\alpha$ *exists* ($\mathfrak{P}''$; unif.) *and* $\mathrm{L}''\alpha$ *exists, then* $\mathrm{L}\alpha$, $\mathrm{L}'\mathrm{L}''\alpha$, $\mathrm{L}''\mathrm{L}'\alpha$ *exist and are equal.*

This theorem follows from 2 and 1, the uniformity implying the sufficient condition of 2.

4_1. *If* $\mathrm{L}''\alpha$ *exists, then in order that* $(\boldsymbol{L}'\mathrm{L}'')\alpha = a$ *it is necessary and sufficient that* $\boldsymbol{L}'\mathrm{L}''\alpha = a$.

It is *necessary.* We readily secure a system $(p'_{ep'}, p''_{ep'} | ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \alpha(p'_{ep'}p'')| \leqq \frac{e}{2} \quad (p''\mathrm{R}''_* p''_{ep'}) \quad (ep'),$$

$$|\alpha(p'_{ep'}p''_{ep'}) - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq \frac{e}{2} \quad (ep').$$

* This proof of the necessity was suggested by Professor W. A. Hurwitz of Cornell University.

Then

$$|a - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq |a - \alpha(p'_{ep'}p''_{ep'})| + |\alpha(p'_{ep'}p''_{ep'}) - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq e \quad (ep').$$

It is *sufficient.* We secure a system $(p'_{ep'}, p''_{ep'}|ep')$ such that

$$p'_{ep'}\mathrm{R}'p', \quad |a - \mathrm{L}''\alpha(p'_{ep'}\diamond)| \leqq \frac{e}{2} \qquad (ep'),$$

$$|\mathrm{L}''\alpha(p'_{ep}\diamond) - \alpha(p'_{ep'}p'')| \leqq \frac{e}{2} \qquad (p''\mathrm{R}''p''_{ep'}) \quad (ep').$$

Then

$$|a - \alpha(p'_{ep'}p'')| \leqq |a - \mathrm{L}''\alpha(p'_{ep'}\diamond)| + |\mathrm{L}''\alpha(p'_{ep'}\diamond) - \alpha(p'_{ep'}p'')| \leqq e \qquad (p''\mathrm{R}''p''_{ep'}) \quad (ep').$$

4_2. *If* $\mathrm{L}'\alpha$ *exists, then in order that* $(\mathrm{L}'\boldsymbol{L}'')\alpha = a$ *it is necessary and sufficient that* $\boldsymbol{L}''\mathrm{L}'\alpha = a$.

5. *If* $\mathrm{L}'\alpha$, $\mathrm{L}''\alpha$, $\mathrm{L}'\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}''\mathrm{L}'\alpha$ *exist and equal* $\mathrm{L}'\mathrm{L}''\alpha$ *it is necessary and sufficient that* $(\boldsymbol{L}'\mathrm{L}'')(\alpha - \mathrm{L}'\alpha) = 0$.

It is *necessary.* Now $\mathrm{L}''(\alpha - \mathrm{L}'\alpha)$ exists equal to $\mathrm{L}''\alpha - \mathrm{L}''\mathrm{L}'\alpha$, since each of thelatter limits exists. Since by § 4, 2 $\boldsymbol{L}'\mathrm{L}''\alpha$ exists uniquely equal to $\mathrm{L}'\mathrm{L}''\alpha$ and obviously $\boldsymbol{L}'\mathrm{L}''\mathrm{L}'\alpha$ exists uniquely equal to $\mathrm{L}''\mathrm{L}'\alpha$, we have

$$\boldsymbol{L}'\mathrm{L}''(\alpha - \mathrm{L}'\alpha) = \mathrm{L}'\mathrm{L}'\alpha - \mathrm{L}''\mathrm{L}'\alpha = 0.$$

Hence by 4_1, $(\boldsymbol{L}'\mathrm{L}'')(\alpha - \mathrm{L}'\alpha) = 0$.

It is *sufficient.* We secure a system $(p'_{1e}, p'_{2e}, p''_e|e)$ such that for every e

$$|\mathrm{L}'\mathrm{L}''\alpha - \mathrm{L}''\alpha(p'\diamond)| \leqq \frac{e}{3} \quad (p'\mathrm{R}'p'_{1e}),$$

$$p'_{2e}\mathrm{R}'p'_{1e}, \quad |\alpha(p'_{2e}p'') - \mathrm{L}'\alpha(\diamond p'')| \leqq \frac{e}{3} \quad (p''\mathrm{R}''_*p''_e).$$

Then

$$|\mathrm{L}'\mathrm{L}''\alpha - \mathrm{L}'\alpha(\diamond p'')| \leqq |\mathrm{L}'\mathrm{L}''\alpha - \mathrm{L}''\alpha(p'_{2e}\diamond)| + |\mathrm{L}''\alpha(p'_{2e}\diamond) - \alpha(p'_{2e}p'')| + |\alpha(p'_{2e}p'') - \mathrm{L}'\alpha(\diamond p'')| \leqq e \quad (p''\mathrm{R}''p''_2) \quad (e).$$

6. *If* $\mathrm{L}'\alpha$, $\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}'\mathrm{L}''\alpha$ *shall exist it is sufficient that* $(\mathrm{L}'\boldsymbol{L}''_0)(\alpha - \mathrm{L}''\alpha) = 0$.

We secure a system $(p'_e, p''_e|e)$ such that for every e

$$|\alpha(p'p'_e) - \mathrm{L}''\alpha(p'\diamond)| \leqq \frac{e}{3} \quad (p'\mathrm{R}'_*p'_e),$$

$$|\alpha(p'p''_e) - \alpha(p'_ep''_e)| \leqq \frac{e}{3} \quad (p'\mathrm{R}'p'_e).$$

Then

$$|\mathrm{L}''\alpha(p'\diamond) - \mathrm{L}''\alpha(p'_e\diamond)| \leqq |\mathrm{L}''\alpha(p'\diamond) - \alpha(p'p''_e)| + |\alpha(p'p''_e) - \alpha(p'_ep''_e)| + |\alpha(p'_ep''_e) - \mathrm{L}''\alpha(p'_e\diamond)| \leqq e \quad (p'\mathrm{R}'p'_e) \quad (e).$$

7. *If* $\mathrm{L}'\alpha$ *and* $\mathrm{L}''\alpha$ *exist, then by* 5, 6 *the following four statements are mutually equivalent*:

(1) $\mathrm{L}'\mathrm{L}''\alpha$ *and* $\mathrm{L}''\mathrm{L}'\alpha$ *exist and are equal*;
(2) $\mathrm{L}'\mathrm{L}''\alpha$ *exists and* $(\boldsymbol{L}'\mathrm{L}'')(\alpha - \mathrm{L}'\alpha) = 0$;
(3) $\mathrm{L}''\mathrm{L}'\alpha$ *exists and* $(\boldsymbol{L}''\mathrm{L}')(\alpha - \mathrm{L}''\alpha) = 0$;
(4) $(\boldsymbol{L}'\mathrm{L}'')(\alpha - \mathrm{L}'\alpha) = 0$, $(\mathrm{L}'\boldsymbol{L}_0'')(\alpha - \mathrm{L}''\alpha) = 0$.

8. *If* $\mathrm{L}'\alpha$, $\mathrm{L}''\alpha$, $\mathrm{L}'\mathrm{L}''\alpha$ *exist, then in order that* $\mathrm{L}''\mathrm{L}'\alpha$ *shall exist and equal* $\mathrm{L}'\mathrm{L}''\alpha$ *it is sufficient that*

$$\boldsymbol{L}'(\alpha - \mathrm{L}'\alpha) = 0 \quad (\mathfrak{P}''; \text{ quasi-unif.}).$$

We secure systems $(p_e'|e)$, $(p'_{ep''}|ep'')$, $(p_e''|e)$ such that for every e

$$|\mathrm{L}''\alpha(p'\diamond) - \mathrm{L}'\mathrm{L}''\alpha| \leqq \frac{e}{3} \quad (p'\mathrm{R}'p_e'),$$

$$[p'_{ep''}|p''] \text{ is finite},$$

$$p'_{ep''}\mathrm{R}'p_e', \quad |\mathrm{L}'\alpha(\diamond p'') - \alpha(p'_{ep''}p'')| \leqq \frac{e}{3} \quad (p''),$$

$$|\alpha(p'_{ep''}p'') - \mathrm{L}''\alpha(p'_{ep''}\diamond)| \leqq \frac{e}{3} \quad (p''\mathrm{R}''p_e'').\dagger$$

Then for every e $p''\mathrm{R}''p_e''$ implies

$$|\mathrm{L}'\alpha(\diamond p'') - \mathrm{L}'\mathrm{L}''\alpha| \leqq |\mathrm{L}'\alpha(\diamond p'') - \alpha(p'_{ep''}p'')| + |\alpha(p'_{ep''}p'') - \mathrm{L}''\alpha(p'_{ep''}\diamond)| + |\mathrm{L}''\alpha(p'_{ep''}\diamond) - \mathrm{L}'\mathrm{L}''\alpha| \leqq e,$$

and accordingly by § 2, 3 $\mathrm{L}''\mathrm{L}'\alpha$ exists equal to $\mathrm{L}'\mathrm{L}''\alpha$, as stated.

§ 8. Lemmas: Revised Formulation of Certain Theorems of Fréchet.*

As foundation for § 8 we have the system

$$\Sigma_3 \equiv (\mathfrak{A};\ \mathfrak{Q};\ \mathfrak{S}^{1.2};\ \mathrm{L}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{Q}\cdot 1.2}),$$

where $\mathfrak{Q} \equiv [q]$ is a class of elements q; $\mathfrak{S} \equiv [s]$ is a class of sequences $s \equiv \{q_n\}$ of elements q_n of $\mathfrak{Q}$ to which belong 1) for every q of $\mathfrak{Q}$ the iterative sequence $\dot{q} \equiv \{q_n = q|n\}$ and 2) every subsequence s_0 of a sequence s of $\mathfrak{S}$; L is a single-valued function on $\mathfrak{S}$ to $\mathfrak{Q}$, associating with every sequence $s \equiv \{q_n\}$ of $\mathfrak{S}$ a definite element q, *the limit of* s, denoted by $\mathrm{L}s$, $\mathrm{L}\{q_n\}$ or $\mathrm{L}_n q_n$, for which 1) $\mathrm{L}\dot{q} = q$ (q) and 2) $\mathrm{L}s_0 = \mathrm{L}s$ for every sequence s of $\mathfrak{S}$ and subsequence s_0 of s.

We denote by Q a non-null subset of the class $\mathfrak{Q}$. For every two sets Q_1 Q_2 the *sum* $\mathrm{Q}_1 + \mathrm{Q}_2$ and the *product* $\mathrm{Q}_1\mathrm{Q}_2$ are respectively the least

† This p_e'' exists since $\mathrm{L}''\alpha(p'\diamond)$ exists for every p' and the elements $p'_{ep''}$ for fixed e are finite in number.

* Fréchet, Sur quelques points du Calcul Fonctionnel, *Rendiconti . . . di Palermo*, Vol. 22 (1906).

common superset and the greatest common subset of the two sets, while the *difference* $Q_1 - Q_2$ is the set of all elements q of Q_1 but not of Q_2; thus $Q_1 + Q_2$ is a set Q while one (but not both) of the sets Q_1Q_2, $Q_1 - Q_2$ may be the null set.

A set Q is *closed* in case for every s (of $\mathfrak{S}$) in Q the limit Ls is of Q. A set Q is *compact* in case every infinite subset of Q contains an s (of $\mathfrak{S}$) consisting of distinct elements. A set Q_1 is *a region relative to a set* Q_2, in notation, a region (Q_2), in case every sequence s in Q_2 with Ls of Q_1 is ultimately in Q_1, i.e., every element of s after a certain one is an element of Q_1; accordingly, every set Q is a region (Q). A set Q is *covered by* a set $[R]$ of regions (Q) in case every q of Q is of some R of $[R]$. A set Q is *enclosable* in case there exists a denumerable set $[R]$ of regions (Q) such that 1) Q is covered by $[R]$ and 2) if an element q of Q is an element of a region (Q), say Q_0, then there exists a region R of $[R]$ contained in Q_0 and containing q.

1. *If* $\{Q_n\}$ *is a sequence of closed compact sets* Q_n, *each containing the following, then the product of all the sets* Q_n *contains at least one element* q.

2. *If a closed compact set* Q *is covered by a denumerable set* $[R_n|n]$ *of regions* (Q), *then the set* Q *is covered by some finite subset of the set* $[R_n|n]$.

For* otherwise, in contradiction to 1, $\{Q - (R_1 + \cdots + R_n)\}$ is a sequence $\{Q_n\}$ of closed compact sets, each containing the following, whose product is the null set, since every element q of Q is of some region R_n and hence not of the corresponding set Q_n.

3. *If an enclosable set* Q *is covered by a set* $[R]$ *of regions* (Q), *then it is covered by some denumerable subset of the set* $[R]$.

Denote by $[R_1]$ a denumerable set of regions (Q) covering the set Q in the sense of the enclosability of Q. An element q lies in a region R and accordingly in a region R_1 contained in R. Thus the set Q is covered by a necessarily denumerable subset $[R_2]$ of the set $[R_1]$ each of which is in a region R. Accordingly the set Q is covered by a denumerable subset $[R_3]$ of the set $[R]$, as stated.

From 2, 3 we have the Heine-Borel-Lebesgue theorem:

4. *If a closed compact enclosable set* Q *is covered by a set* $[R]$ *of regions* (Q), *then the set* Q *is covered by some finite subset of the set* $[R]$.

A function β on Q to $\mathfrak{A}$ is *continuous at an element* q of Q, in case $L_n\beta(q_n) = \beta(q)$ for every sequence $\{q_n\}$ in Q with $L_nq_n = q$; and it is *continuous on* Q, in case it is continuous at every q of Q.

§ 9. Composite Range. Continuity.

The theorems of this section are with reference to the foundation.

$$\Sigma_4 = (\mathfrak{A};\ \mathfrak{P};\ R^{\text{on } \mathfrak{P}\cdot TQ};\ \mathfrak{Q};\ \mathfrak{S}^{1.2};\ L^{\text{on } \mathfrak{S} \text{ to } \mathfrak{Q}\cdot 1.2}),$$

* Compare Hausdorff, *Grundzüge der Mengenlehre*, p. 272.

where the notations have the same meanings as in §§ 1, 6, 8.

We consider a subset Q of $\mathfrak{Q}$ and a function φ on $\mathfrak{P}$Q to $\mathfrak{A}$.

1. *If the function $\varphi(p\diamond)$ is continuous on the set* Q *for every* p, *and* $\mathrm{L}\varphi(\diamond q)$ *exists for every* q *of* Q, *then in order that* $\mathrm{L}\varphi$ *shall be continuous on* Q *it is sufficient that*

$$\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0 \quad (\mathrm{Q};\ \text{quasi-unif.}).$$

Consider an element q of Q and a sequence $\{q_n\}$ of Q with $\mathrm{L}_n q_n = q$. We are to prove that $\mathrm{L}_n\mathrm{L}\varphi(\diamond q_n)$ exists and is equal to $\mathrm{L}\varphi(\diamond q)$. This is a consequence of § 7, 8. For $\mathrm{L}\varphi(\diamond q_n)$ exists for every n; $\mathrm{L}_n\varphi(pq_n)$ exists for every p; $\mathrm{LL}_n\varphi(\diamond q_n)$, quâ $\mathrm{L}\varphi(\diamond q)$, exists, and the sufficient condition stated implies

$$\boldsymbol{L}[\varphi(\diamond q_n) - \mathrm{L}\varphi(\diamond q_n)] = 0 \quad ([n];\ \text{quasi-unif.}).$$

2. *If the set* Q *is compact and closed, and the function $\varphi(p\diamond)$ is continuous on* Q *for every* p, $\mathrm{L}\varphi(\diamond q)$ *exists for every* q *of* Q, *and* $\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0$ (Q; semi-unif.), *then in order that* $\mathrm{L}\varphi$ *shall be continuous on* Q, *it is necessary that* $\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0$ (Q; quasi-unif.).

There is a set $(p_{1epq}|epq^{\text{ of Q}})$ such that

1) $[p_{1epq}|q^{\text{ of Q}}]$ is denumerable (ep),

2) $p_{1epq}\mathrm{R}p$, $|\varphi(p_{1epq}q) - \mathrm{L}\varphi(\diamond q)| \leqq \frac{e}{2} < e$ $(epq^{\text{ of Q}})$.

For every $(epq^{\text{ of Q}})$ let Q_{epq} denote the set of all elements q' of Q for which

$$|\varphi(p_{1epq}q') - \mathrm{L}\varphi(\diamond q')| < e.$$

Every set Q_{epq} is a region (Q). Further for every (ep) the set $[\mathrm{Q}_{epq}|q^{\text{ of Q}}]$ is a denumerable set of regions (Q) covering Q. Hence by § 8, 2 there is for every (ep) a finite number of them

$$[\mathrm{Q}_{epq_1}, \mathrm{Q}_{epq_2}, \cdots]$$

which cover Q; for every q of Q set $p_{epq} \equiv p_{1epq_i}$ where i is the smallest integer such that Q_{epq_i} contains q. Then the system $(p_{epq}|epq^{\text{ of Q}})$ is effective in proof of the desired quasi-uniformity.

3. *If the set* Q *is enclosable and the function $\varphi(p\diamond)$ is continuous on* Q *for every* p *and* $\mathrm{L}\varphi(\diamond q)$ *exists for every* q *of* Q, *then if* $\mathrm{L}\alpha$ *is continuous on* Q *it is true that*

$$\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0 \quad (\mathrm{Q};\ \text{semi-unif.}).$$

For since $\mathrm{L}(\varphi - \mathrm{L}\varphi) = \mathrm{L}\varphi - \mathrm{LL}\varphi = \mathrm{L}\varphi - \mathrm{L}\varphi = 0$ (Q), we have $\boldsymbol{L}(\varphi - \mathrm{L}\varphi) = 0$ (Q); hence there exists a set $(p_{1epq}|epq^{\text{ of Q}})$ such that

$$p_{1epq}\mathrm{R}p, \quad |\varphi(p_{1epq}q) - \mathrm{L}\varphi(\diamond q)| \leqq \frac{e}{2} < e\ (epq^{\text{ of Q}}).$$

For every $(epq^{\text{of Q}})$ let Q_{epq} denote the set of all elements q' of Q for which $|\varphi(p_{1epq}q') - L\varphi(\diamond q')| < e$. Then every Q_{epq} is a region (Q). Further for every (ep) $[Q_{epq}|q^{\text{of Q}}]$ is a class of regions (Q) covering Q. Then by § 8, 3 there is a denumerable subset $[Q_{epq_1}, Q_{epq_2}, \cdots]$ of $[Q_{epq}|q^{\text{of Q}}]$ covering Q; for every q of Q set $p_{epq} \equiv p_{epq_i}$, where i is the smallest integer such that Q_{epq_i} contains q. Then the system $(p_{epq}|epq^{\text{of Q}})$ is effective in proof of the desired semi-uniformity.

From 1, 2, 3 we have the theorem of Arzelà:*

4. *If $\varphi(p\diamond)$ is continuous on* Q *for every* p *and* $L\varphi(\diamond q)$ *exists for every* q *of* Q, *then in order that* $L\varphi$ *shall be continuous on* Q *it is sufficient that* $\boldsymbol{L}(\varphi - L\varphi) = 0$ (Q; quasi-unif.). *If* Q *is compact, closed and enclosable, then that condition is necessary.*

5. *If $\varphi(\diamond q)$ is monotone decreasing for every* q *of* Q, *it is true that*

(1) *if* $\boldsymbol{L}\varphi$ *exists* (Q), *then* $L\varphi$ *exists* (Q);

(2) *if* $\boldsymbol{L}\varphi$ *exists* (Q; semi-unif.), *then* $L\varphi$ *exists* (Q; semi-unif.).

(3) *if* $\boldsymbol{L}\varphi$ *exists* (Q; quasi-unif.), *then* $L\varphi$ *exists* (Q; unif.).

This is easily proved. From 4, 5_3 we have at once the theorem of Dini:

6. *If* Q *is closed, compact and enclosable, and the function $\varphi(p\diamond)$ is continuous on* Q *for every* p *and $\varphi(\diamond q)$ is monotone decreasing for every* q *of* Q, *and* $L\varphi = 0$, *then* $L\varphi = 0$ (Q; unif.).

THE UNIVERSITY OF CHICAGO,
THE UNIVERSITY OF THE PHILIPPINES,
October 30, 1921.

* Arzelà, "Sulle Serie di Funzioni," *Rend. di Bologna*, Ser. V, Vol. 8, 1899.

SUBSTITUTION GROUPS WHOSE CYCLES OF THE SAME ORDER CONTAIN A GIVEN NUMBER OF LETTERS.

By G. A. Miller.

Let G be any transitive or intransitive substitution group, and let $a_1a_2\cdots a_k$ be any one of the cycles found in one or more of the substitutions of G. If this cycle and all of its conjugates under G are counted once for every substitution of G in which they appear, then the total number of letters in all of these cycles is exactly the order of G, since such a cycle is transformed into itself only by its powers by means of substitutions restricted to its own letters.* Hence the total number of letters in all the cycles of order k contained in G may be found as follows: Select a set of cycles of order k composed of all the different cycles of this order found in G. Let λ be the number of complete sets of conjugates under G contained in this set. The total number of letters in all the cycles of order k found in G is then λg, g being the order of G.

In particular, the holomorph of the cyclic group of order p, p being any prime number, contains $p-1$ cycles of order p. Since these cycles form a single set of conjugates under this holomorph $\lambda = 1$ in this case, and hence the order of this holomorph is $p(p-1)$, as is also otherwise evident. As another very elementary illustration it may be noted that every intransitive group of order p^m which has only transitive constituents of degree p involves only invariant cycles of order p. The number of the distinct cycles is $p-1$ times n/p, n being the degree of the group. Hence $\lambda = n(p-1)/p$ in this case and the total number of letters in all the substitutions of G is $n(p-1)p^{m-1}$ as is also evident from the fact that the average number of letters in all the substitutions of an intransitive group of degree n which has k transitive constituents is $n-k$.

A necessary and sufficient condition that a group G of degree n which has k systems of intransitivity has the property that the total number of letters in all of its cycles of the same order is exactly g for every cycle which appears in these substitutions is that G contains cycles of each of the orders $2, 3, \cdots, n-k+1$. This is clearly impossible when G is intransitive since the degree of each of these transitive constituents would be less than $n-k+1$. Moreover, it is clear that the symmetric group of degree n involves cycles of each of the orders $2, 3, \cdots, n$, and hence this group has the property that the total number of letters in all the cycles of the same

* E. Netto, *Journal für die reine und angewandte Mathematik*, Vol. 103 (1888), p. 323.

order is $n!$ for every cycle which appears in the group. It is easy to verify that the well-known triply transitive group of degree 6 and of order 120 has the same property.

To prove that this is the only non-symmetric group which has this property it may first be noted that if the degree of such a group would exceed 7 the group would involve a cycle of prime order p, where p satisfies the condition $n/2 < p \leqq n - 3$, according to a well-known theorem due to Tshebychef. Hence such a group must be primitive, and it cannot involve a cyclical substitution of order p without being either alternating or symmetric.* As the group could not be alternating since in the alternating group of degree n there cannot be cycles of both the orders n and $n - 1$, it must be symmetric.

It remains therefore only to prove that there is no transitive group of degree n less than 8, besides the triply transitive group of degree 6 to which we have already referred, which has the property that its substitutions involve cycles of each of the orders $2, 3, \cdots, n$. As these groups are well known this proof may be regarded as obvious, since it implies at most an examination of a few lists of substitution groups of low degrees.

To the many interesting known properties of the triply transitive group of degree 6, which was considered at length by A. L. Cauchy in volume 22 (1846) of the *Comptes Rendus*, and had been studied earlier by C. Hermite, we have here added one which all symmetric groups possess but which no other non-symmetric group possesses. This property therefore belongs to this group both when it is represented on 5 letters and when it is represented on 6 letters. In all other cases the symmetric group of degree n loses this property when it is represented on more than n letters.

The alternating group of degree n involves cycles of each of the orders $2, 3, \cdots, n$ except of order n, when n is even, or of order $n - 1$, when n is odd. Hence it results directly that the total number of letters in all the cycles of the same order contained in the alternating group is equal to the order of the group with the exception that this total is equal to twice the order of the group in the case of the largest cycle found in such a group. This follows directly from the fact that these largest cycles constitute two complete sets of conjugates under the alternating group. A necessary and sufficient condition that the holomorph of a cyclic group, written as a regular group, is composed of positive substitutions is that the orders of all the Sylow subgroups of this cyclic group are even powers of odd prime numbers.* This is evidently also a necessary and sufficient condition that

* G. A. Miller, *Bulletin of the American Mathematical Society*, Vol. 4 (1898), p. 141.

* This fundamental theorem relating to the holomorph of a cyclic group is not found in my article on this subject, *Transactions of the American Mathematical Society*, Vol. 4 (1903), p. 153.

all the generators of a cyclic group appear in one of the two complete sets of conjugates noted above.

Besides the alternating groups there are various others which have the property that the total number of letters in all the cycles of the same order is equal to the order of the group with the exception that for one such order this number is twice the order of the group. If such a group is transitive it must involve cycles of each of the orders 2, 3, $\cdots$, n except for one such order. This condition is evidently sufficient as well as necessary. It is easily seen that the degree of such a transitive group could not exceed 15 since there are at least two distinct prime numbers which satisfy the condition $n/2 < p \leqq n - 3$ whenever n is a positive integer greater than 15. This fact can easily be verified by means of tables of primes extending to one million, and for larger values of n it results directly from known formulas. Cf. E. Landau, *Primzahlen*, Vol. 1 (1909), p. 91. In fact, from such formulas it results also that at least two such primes always exist when n exceeds a much smaller number than one million.

From the existence of at least two such primes it results directly, just as in the case of symmetric groups, that if the degree of a transitive group exceeds 15 it cannot have cycles of each of the orders 2, 3, $\cdots$, n, except one, without being alternating. It is easy to verify that no non-alternating group whose degree exceeds 8 satisfies these conditions and that the group of isomorphisms of the simple group of order 168 is the only transitive group of degree 8 which is not alternating but satisfies the conditions in question. In this group of order 336, just as in the alternating groups, the number of letters in all of the cycles of the same order, except the largest one, is equal to the order of the group, while the number of letters in the cycles of order 8 is equal to 672.

There is no transitive group of degree 7 which satisfies the conditions in question but there are two such groups of degree 6, viz., the largest groups which contain 2 or 3 systems of imprimitivity. These groups are of orders 72 and 48 respectively, and the number of letters in all their cycles of lowest order is twice the order of the group while the number of letters in all of their cycles of each other order is equal to the order of the group. On 5 letters there is clearly only one such non-alternating transitive group, viz., the group of order 20, while on 4 letters the cyclic group and the octic group constitute the only instances. In the cyclic group and the group of order 20 the number of letters in the cycles of order four is equal to twice the order of the group while in the octic group this is the case as regards the cycles of lowest order.

Hence there are just six transitive groups which are non-alternating but have the property that the total number of letters in all the cycles of each

order save one is the order of the group while for this one the total number of letters is twice the order of the group. Three of these have in common with all the alternating groups the property that the cycles in which the largest number of letters appear are also the largest cycles, while in the other three the largest number of letters appear in the smallest cycles.

If in a constituent of an intransitive group the total number of letters found in all of its cycles of the same order is k times the order of this constituent, then the total number of letters found in these cycles for all the substitutions of the entire group is also equal to k times the order of this group. In particular, if each of k constituents of an intransitive group contains a cycle of the same order, then the total number of letters in all the cycles of this order contained in the group is at least k times the order of the group. Hence it results that if an intransitive group has the property that the total number of letters found in its cycles of the same order, except one, is equal to the order of the group and for this one it is equal to twice this order, then no more than two of its transitive constituents can involve cycles of the same order. If two such constituents involve cycles of the same order, the total number of letters in all the cycles of the same order found in these constituents must be the order of the constituent.

It is evident that the intransitive groups which have for one constituent the regular group of order 2 and for the other any symmetric group, or the triply transitive group of degree 6, satisfy the condition that the total number of letters found in their cycles of order 2 is twice the order of the group while the total number of letters in all of the other cycles of the same order is equal to the order of the group. Moreover, these are the only intransitive groups in which at least two transitive constituents have cycles of the same order and which, moreover, satisfy the condition that the total number of letters found in the cycles of one order is twice the order of the group while the total number of letters found in all the cycles of every other order is equal to the order of the group. If no two transitive constituents have cycles of the same order, the orders of these constituents must be relatively prime.

As the orders of all the transitive groups which have the property that either the total number of letters in the cycles of the same order is equal to the order of the group for every order of such a cycle, or this is true of all cycles except those of one order in which the total number of letters involved is twice the order of the group, are even with the exception of the alternating group of degree 3, it remains therefore only to consider the intransitive groups having for one constituent this alternating group. The other constituent is evidently the symmetric group of order 2, and hence the intransitive cyclic group of degree 5 and order 6 is the only intransitive

group which has the property that the total number of letters in every cycle of the same order except one is equal to the order of the group and that the number of letters in the cycles of this particular order is equal to twice the order of the group while the orders of the transitive constituents are relatively prime.

If no two of the complete set of conjugates of a given cycle of a transitive group have a common letter, then these conjugates constitute a substitution which is invariant under the transitive group. Moreover, if a substitution is invariant under a transitive group, its subgroup which is composed of all its substitutions omitting one letter must omit more than one letter, and hence the group must *involve* a cycle whose conjugates under the group have no common letter. That is, *a necessary and sufficient condition that there is a substitution, besides the identity, which is commutative with every substitution of a transitive group is that the group involves a cycle such that no two of the conjugates of this cycle under the group have a common letter.* The total number of substitutions which are commutative with every substitution of a transitive group is therefore equal to the number of its different cycles which involve the same letter and have the property that no two of the cycles in any one of the complete sets of conjugates to which they separately belong have a common letter.

As the total number of letters found in identical cycles and all their conjugates is equal to the order of the group, it results directly that the number of cycles which are identical with a cycle having the property that no two of its conjugates involve the same letter is equal to the order of the group divided by its degree. In particular, the number of such cycles which are identical is an invariant of the substitution group. A necessary and sufficient condition that a transitive group is regular is that every one of its cycles has the property that no two of its conjugates under the group have a common letter. The $g - 1$ complete sets of conjugates of the cycles of a regular group G of order g give rise to $g - 1$ substitutions which are separately commutative with every substitution of G. The fact that these $g - 1$ substitutions together with the identity constitute a group follows directly from the facts that all the substitutions which are commutative with every substitution of G must constitute a group and that each such substitution besides the identity must be of degree g. We thus have a new proof of the fact that all the substitutions which are commutative with each substitution of a regular group of order g constitute a regular group of this order.

Since every transitive group whose order is of the form p^m, p being a prime number, involves invariant substitutions, it results that it involves cycles such that no two of their conjugates under the group involve a

common letter. When the group is non-regular each such cycle must appear in more than one substitution. In fact, the number of substitutions in which it appears is always equal to the order of the group divided by its degree even when the order of the group is not a power of a prime. In particular, when the degree of a group of order p^m is p^2, $m > 2$, its only cycles which appear in more than one substitution are those which are found in its invariant subgroup of order p. That is, *in every transitive group of degree p^2 and of order p^m, $m > 2$, the total number of letters found in the conjugates of every cycle which does not appear in the invariant subgroup of order p is equal to the order of the group.*

If a transitive group of degree n is regular or of class $n - 1$, it evidently cannot involve two substitutions which contain the same cycle. It is not difficult to prove that every other transitive group contains such substitutions. To prove this theorem, let G be any transitive group of degree n which involves at least one substitution s whose degree does not exceed $n - 2$. If G is at least doubly transitive, it must involve a cycle of order 2 which does not involve any of the letters of s. The subgroup generated by s and a substitution involving this cycle will clearly have this cycle in one half of its substitutions, and as the order of this subgroup exceeds 2 it may be assumed in what follows that G is simply transitive. It will also be convenient to assume that G is a group of lowest possible degree which does not involve two substitutions containing the same cycle but involves at least one substitution whose degree does not exceed $n - 2$.

The subgroup G_1 composed of all the substitutions of G which omit a given letter is of degree $n - \alpha$. If $\alpha > 1$, it is well known that G_1 is invariant under an intransitive group which has one transitive constituent of degree α. This intransitive group evidently involves at least two substitutions containing the same cycle. Hence it may be assumed that $\alpha = 1$ and that G_1 is formed by a simple isomorphism between transitive groups of which at least one is of a class one less than its degree while the others, if any, are regular. If a transitive group of degree k and of class $k - 1$ can be represented in more than one way as a transitive group whose degree is exactly one unit larger than its class, every invariant regular subgroup must include all the subgroups which appear in the regular form when the group is represented on a smaller number of letters. This follows directly from the fact that in a transitive group of degree k and of class $k - 1$ every substitution of degree $k - 1$ is transformed into k distinct substitutions by the substitutions of the regular subgroup of order k and this subgroup of order k must be found in every invariant subgroup which involves a substitution of degree $k - 1$.

The substitution s may be supposed to be found in G_1. If the group generated by s is transformed into itself by a substitution of G which is not

found in G_1, this substitution and s will clearly generate an intransitive group having a constituent whose order is less than the order of this intransitive group and therefore a cycle which appears in more than one of its substitutions. Hence it may be supposed that s is transformed under G into a substitution of G_1 with which it is not conjugate under G_1. As this substitution may be supposed to involve the same letters as s does, it results that the transforming substitution and s again generate an intransitive group involving a constituent whose order is less than the order of this intransitive group. Hence it has been proved that *a necessary and sufficient condition that a transitive group of degree n contains a complete set of distinct conjugate cycles whose total number of letters is less than the order of the group is that the class of the transitive group is less than $n - 1$.*

If a substitution group of degree n involves cycles of order n or of order $n - 1$, the total number of letters in a complete set of conjugates of such a cycle is always equal to the order of the group since such a cycle cannot appear in two different substitutions. This is a special case of the evident theorem that if a group of degree n involves a cycle of order $n - \alpha$ but no substitution on α or less than α letters, then the total number of letters in all the conjugates of this cycle must equal the order of the group. In particular, if a primitive substitution group of degree n involves a complete set of conjugate cycles of order $n - 2$ such that the total number of letters in all the cycles of the set is less than the order of the group, the group must be the symmetric group of degree n, and if an imprimitive group of degree n involves such a complete set of conjugates, n must be even and the group must involve the abelian subgroup which has for its independent generators $n/2$ transpositions.

In the preceding paragraph there appears a new definition of the symmetric group. As another such definition we give the following: The symmetric group of degree n is the transitive group of this degree in which each cycle of order 2 appears in $(n - 2)!$ substitutions. It may also be noted that if a multiply transitive group of degree n is not of class $n - 1$, then each of its cycles of order 2 must be found in more than one substitution. In fact, each such cycle can be transformed into every other such cycle since the group is at least two times transitive. Some such cycle must appear in more than one substitution, since the cycle of order 2 which involves the letter a, for instance, must involve another letter which does not appear in at least two of the substitutions of the subgroup composed of all the substitutions which omit a. Hence the theorem: *Every cycle of order 2 found in a multiply transitive group appears in more than one substitution whenever the class of this group is not one less than its degree.*

UNIVERSITY OF ILLINOIS

BOUNDARY VALUE AND EXPANSION PROBLEMS: OSCILLATION, COMPARISON AND EXPANSION THEOREMS.*

By R. D. Carmichael.

1. *Algebraic Oscillation and Comparison Theorems.*—On any convenient horizontal straight line segment, say the points s such that $a \leqq s \leqq b$, let us erect n perpendiculars two of which are at the ends of the segment while the other $n - 2$ are evenly or unevenly distributed on the interior of the segment. Let these be marked from left to right by the numbers $1, 2, \cdots, n$; and consider them as analogous to the n coördinate axes of a space of n dimensions. Let the greatest distance between two consecutive axes be called the norm of the system of axes. Having given the set of real constants $u_1, u_2, \cdots, u_n$, let us take a point on the ith axis at a distance $|u_i|$ from the original segment and above it or below it according as u_i is positive or negative. Having done this for each value i of the set $1, 2, \cdots, n$, join by straight line segments the point on each interior axis to the points on the two adjacent axes. We thus obtain a broken line which we shall call the graphic representation of the point $(u_1, u_2, \cdots, u_n)$ in space of n dimensions or of the set of constants $u_1, u_2, \cdots, u_n$. This broken line is the graph of a continuous function $u(s)$ of the real variable s on the interval $a \leqq s \leqq b$. We shall say that this function $u(s)$ is obtained from the set of constants $u_1, u_2, \cdots, u_n$ by *linear interpolation* with respect to the given n axes. The zeros of $u(s)$ we shall call the *zeros* of the set of constants with respect to the given system of coördinates.

Let us consider the system of n equations

$$\sum_{j=1}^{n+2} a_{ij}x_j = 0, \qquad i = 1, 2, \cdots, n, \tag{1.1}$$

in the $n + 2$ unknown quantities $x_1, x_2, \cdots, x_{n+2}$, the matrix of coefficients of this system being of rank n. Let D_i denote the determinant of the matrix obtained from the matrix of coefficients in (1.1) by striking out the ith and $(i + 1)$th columns. We then have the following fundamental theorem:†

Theorem I. *Let D_i for a given range R of consecutive values of the integer i be of one sign and let I denote the interval of the s-axis corresponding to this range of i in the sense of the first paragraph above. Let u_i and v_i be any two linearly independent solutions of the system* (1.1) *the matrix of whose coefficients is of rank n; and let these solutions be extended, by the method of*

* Presented to the American Mathematical Society.

† *American Journal of Mathematics*, 43 (1921): 69–101; see p. 84.

linear interpolation employed above, to the functions $u(s)$ and $v(s)$. Then on the interval I the zeros of $u(s)$ and $v(s)$ separate each other; that is, between any two consecutive zeros (on I) of one of these functions there is one and just one zero of the other function.

On examining the proof of this theorem, in the article cited, it is seen that the only use made of the hypothesis on D_i is in showing that ω_i,

$$\omega_i = \begin{vmatrix} u_i & v_i \\ u_{i+1} & v_{i+1} \end{vmatrix},$$

is of one sign in I, so that the theorem might be restated with D_i replaced by ω_i in the first line. The theorem as first stated is in the more useful form for suggesting analogous theorems in the transcendental cases; but the second form will be found more suggestive for proofs.

The foregoing theorem is analogous to the Sturmian theorem of oscillation for homogeneous linear differential equations of the second order and indeed reduces to that theorem by a certain limiting process. The object of this paper is (*a*) to derive (§ 1) the algebraic theorems which are analogous to the Sturmian theorems of comparison for homogeneous linear differential equations of the second order and of which the latter are limiting forms, (*b*) to obtain (§ 2) theorems of oscillation for differential equations of order n and for certain functional equations including difference and q-difference equations, (*c*) to derive (§ 3) corresponding theorems of comparisons by aid of the named algebraic theorems of comparison, (*d*) to point out (§ 4) a certain generalization of boundary conditions for expansion problems, and (*e*) to indicate (§ 5) the character of certain expansion problems for q-difference and integro-q-difference equations.

In what follows in this section we shall assume that the notation in (1.1) is so chosen that the determinant D_1 is different from zero. If the equations are taken in a suitable order (and we shall suppose them so written already), it is obviously possible to combine them into a new system having the same solutions x in such way that the new coefficients a_{ij} have the value zero when $j > i + 2$ and such that every $a_{i,i+2}$ is different from zero; and we reduce the latter to unity by dividing both members of the ith equation by $a_{i,i+2}$. We shall now suppose further that the original, and hence the new, matrix $||a_{ij}||$ has the property that the determinants of orders $1, 2, \cdots, n$ in its upper left-hand corner are all different from zero in value. Then it is possible to make further combinations of equations in the system so as to arrive at a new system with the same solutions x and of such sort that the new coefficients a_{ij} are zero when $j < i$. Then, changing the order of terms in the equation, we have a system in the form

$$x_{i+2} + \alpha_i x_{i+1} + \beta_i x_i = 0, \qquad i = 1, 2, \cdots, n, \tag{1.2}$$

where α_i and β_i are determinate functions of the original coefficients a_{ij}.

On writing $x_i = y_i u_i$, the foregoing equation reduces to the following:

$$y_{i+2}u_{i+2} + \alpha_i y_{i+1}u_{i+1} + \beta_i y_i u_i = 0. \tag{1.3}$$

We set $y_{i+2} = \beta_i y_i$. If y_1 and y_2 are any two numbers different from zero, we have

$$y_{2i+1} = \beta_{2i-1}\beta_{2i-3}\cdots\beta_3\beta_1 y_1, \qquad y_{2i} = \beta_{2i-2}\beta_{2i-4}\cdots\beta_4\beta_2 y_2.$$

If β_i is positive for all i and if y_1 and y_2 are positive, it is clear that y_i is always positive and hence that u_i and x_i have always the same sign. Therefore the functions $x(s)$ and $u(s)$, obtained by linear interpolation from x_i and u_i with respect to a given system of coördinates, have their zeros on the same intervals of the coördinate system (though not necessarily at the same points). Equation (1.3) may be written in the form

$$u_{i+2} + \varphi_i u_{i+1} + u_i = 0, \qquad \varphi_i = \alpha_i \frac{y_{i+1}}{y_{i+2}}, \tag{1.4}$$

where φ_i is a determinate function of the a_{ij} in (1.1). From the principal properties of the distribution of the zeros of $u(s)$ one knows the principal properties of the distribution of the zeros of $x(s)$ provided that β_i is positive for each value of i.

A second equation of the form (1.1), under such hypotheses as we have just employed, would reduce to the normal form

$$v_{i+2} + \psi_i v_{i+1} + v_i = 0. \tag{1.5}$$

Comparison theorems for the distribution of the zeros of the functions $u(s)$ and $v(s)$, obtained from the constants u_i and v_i by linear interpolation, yield corresponding theorems for the two original systems of form (1.1). We shall derive and state the results only for the normal forms (1.4) and (1.5).

Multiplying (1.4) member by member by v_{i+1} and (1.5) by $-u_{i+1}$ and adding, we have a result which may be put in the form

$$\Delta\{u_{i+1}v_i - u_i v_{i+1}\} + (\varphi_i - \psi_i)u_{i+1}v_{i+1} = 0;$$

whence, on summing as to i from μ to m, it follows that

$$(u_{m+2}v_{m+1} - u_{m+1}v_{m+2}) - (u_{\mu+1}v_\mu - u_\mu v_{\mu+1}) + \sum_{i=\mu}^{m} (\varphi_i - \psi_i)u_{i+1}v_{i+1} = 0, \tag{1.6}$$

where μ and m are any two numbers of the set $1, 2, \cdots, n$ such that $\mu \leqq m$.

If solutions u_i and v_i of (1.4) and (1.5), respectively, interpolate into functions $u(s)$ and $v(s)$ having a common zero, say on the μth interval, and if $u(s)$ has a zero on an interval to the right of the μth, say on the

$(m+1)$th interval, m being chosen as small as possible, we may show that $v(s)$ has a zero between these two zeros of $u(s)$ provided that $\varphi_i \leqq \psi_i$, $i = \mu, \mu+1, \cdots, m$, the equality sign not holding for all these values. Without loss of generality in argument we take $u(s)$ and $v(s)$ to be positive each on the interval from its zero on the μth interval to its next zero to the right. We shall prove the statement in consideration by showing that we are led to a contradiction if we suppose that $v(s)$ is nowhere negative between the two consecutive zeros of $u(s)$. We have $u_{\mu+1}v_\mu - u_\mu v_{\mu+1} = 0$, since $u(s)$ and $v(s)$ have a common zero on the μth interval. Then from (1.6) we see that we now have

$$u_{m+2}v_{m+1} - u_{m+1}v_{m+2} = \sum_{i=\mu}^{m} (\psi_i - \varphi_i)u_{i+1}v_{i+1} > 0.$$

This requires that $v(s)$ shall have a zero on the $(m+1)$th interval and to the left of that of $u(s)$, so that $v(s)$ is negative near the right-hand end of the interval between the named consecutive zeros of $u(s)$, contrary to hypothesis. Hence $v(s)$ has a zero between these two consecutive zeros of $u(s)$.

If we take any solution $\bar{v}_i$ of (1.5) linearly independent of v_i, then the corresponding function $\bar{v}(s)$ has a zero between two consecutive zeros of $v(s)$ by theorem I. Hence $\bar{v}(s)$ has a zero on the interior of the interval between two consecutive zeros of $u(s)$. Combining this result with that in theorem I we have the following fundamental theorem of comparison:*

THEOREM II. *Let u_i and v_i be solutions of* (1.4) *and* (1.5), *respectively, and let $u(s)$ and $v(s)$ denote the functions into which they interpolate linearly with respect to a given system of coördinates. If $u(s)$ has consecutive zeros on the μth and $(m+1)$th intervals, $\mu < m$, then $v(s)$ has a zero between these zeros of $u(s)$ provided that either*

(a) $\varphi_i \leqq \psi_i$, $i = \mu, \mu+1, \cdots, m$, the equality sign not holding for all these values; or,

(b) $\varphi_i = \psi_i$, $i = \mu, \mu+1, \cdots, m$, and the sets of constants u_i and v_i, for $i = \mu, \mu+1, \cdots, m$, are linearly independent.

By means of this theorem we can readily establish other properties of comparison for $u(s)$ and $v(s)$. *Suppose that $u_1 \neq 0$, $v_1 \neq 0$, $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, \nu$, and that $u_2/u_1 > v_2/v_1$. Then, if $u(s)$ has k zeros on the first ν intervals of the coördinate system, $v(s)$ has at least k zeros on these intervals and the jth of these zeros (in increasing order) of $v(s)$ is to the left of the jth one of $u(s)$.* In view of theorem II it is sufficient to prove that the first zero of $v(s)$ is to the left of the first zero of $u(s)$. Let the $(m+1)$th interva be the one containing the first zero of $u(s)$. If $m = 0$ we employ the

* Compare the related theorem due to M. B. Porter, *Annals of Mathematics* (2), 3 (1902), p. 65.

relation $u_2/u_1 > v_2/v_1$ to show readily that $v(s)$ has a zero to the left of the first zero of $u(s)$. If $m > 0$ we employ (1.6) for $\mu = 1$. Either $v(s)$ vanishes to the left of the first zero of $u(s)$ or, on taking u_1 and v_1 positive (as we may do in proof without loss of generality), we have $u_{m+2}v_{m+1} - u_{m+1}v_{m+2} > 0$, so that $v(s)$ has a zero on the $(m+1)$th interval to the left of that of $u(s)$ on this interval. This establishes the property in consideration.

Let u_1, v_1, u_{k+1}, v_{k+1} be all different from zero and let $u_2/u_1 > v_2/v_1$. Let $u(s)$ and $v(s)$ have the same number (which may be zero) of roots on the first k intervals. Then we have

$$\frac{u_{k+2}}{u_{k+1}} > \frac{v_{k+2}}{v_{k+1}}, \tag{1.7}$$

provided that $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, k$. From the result in the foregoing paragraph it follows that the rightmost root of $u(s)$ on these k intervals is to the right of that of $v(s)$, if either has a root on these intervals. Taking first the case in which they have at least one such root, let μ denote the number of the interval containing the rightmost root of $u(s)$ not at its right extremity. Without loss of generality in argument, we take $u_{\mu+1}$ and $v_{\mu+1}$ to be equal and positive. Then $v_\mu > u_\mu$ so that $u_{\mu+1}v_\mu > u_\mu v_{\mu+1}$. In (1.6) we take $m = k$. Then we have $u_{k+2}v_{k+1} > u_{k+1}v_{k+2}$, which reduces to (1.7). If $u(s)$ and $v(s)$ have no root on the k intervals, we take u_1 and v_1 positive (as we may in proof without loss of generality). We employ (1.6) with $\mu = 1$ and $m = k$. We have again $u_{k+2}v_{k+1} > u_{k+1}v_{k+2}$, whence (1.7) follows at once.

In like manner certain other similar results may be obtained from theorem II by modifying the form of certain of the inequalities in hypothesis and conclusion.

2. *Transcendental Oscillation Theorems.*—It is well known* that various types of transcendental equations may be realized as limiting cases of algebraic systems. For certain of these we shall now prove the propositions into which theorem I of § 1 passes under the appropriate limiting processes. For a homogeneous linear differential equation of the second order, $y'' + py' + qy = 0$, the result is classic in the Sturm theory: the zeros of any two linearly independent real solutions separate each other throughout any interval in which p and q are real-valued single-valued continuous functions of the real variable x.

Let us consider the difference equation

$$L(x)u(x) + M(x)u(x+1) + N(x)u(x+2) = 0 \tag{2.1}$$

in which all the indicated functions are real-valued single-valued continuous

* For references see a paper entitled "Algebraic Guides to Transcendental Problems" in *The Bulletin of the American Mathematical Society*, (2) 28 (1922): pp. 179–102.

functions of the real variable x for $x \geqq \alpha$, and $L(x)$ and $N(x)$ are both of one and the same sign for $x \geqq \alpha$. Let $u_1(x)$ and $u_2(x)$ be a fundamental system of solutions of this equation. Then, if we write

$$\omega(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1(x+1) & u_2(x+1) \end{vmatrix}, \tag{2.2}$$

we have

$$\omega(x+1) = \begin{vmatrix} u_1(x+1) & u_2(x+1) \\ u_1(x+2) & u_2(x+2) \end{vmatrix} = \frac{L(x)}{N(x)}\omega(x),$$

the last member being gotten from the second last on replacing $u_i(x+2)$ $(i = 1, 2)$ by its value in terms of $u_i(x)$ and $u_i(x+1)$ gotten from the fact that $u_i(x)$ satisfies (2.1). It follows from this that the numbers $\omega(a)$, $\omega(a+1)$, $\omega(a+2)$, $\cdots$ are all of one sign (and hence not zero) if a is a real number not less than α and such that $\omega(a) \neq 0$.

Let us now consider the set of constants $u_i(a)$, $u_i(a+1)$, $u_i(a+2)$, $\cdots$ and let us interpolate them linearly into the function $\bar{u}_i(x)$ with respect to a system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, $a+1$, $a+2$, $\cdots$. Then from theorem I of § 1 and the remark following it we see that the zeros of the functions $\bar{u}_1(x)$ and $\bar{u}_2(x)$ separate each other throughout the range $a \leqq x < \infty$. Let the zeros of $\bar{u}_i(x)$ on the range $a \leqq x < \infty$ be called the *characteristic points* x for the function $u_i(x)$ with respect to the point a. Then a characteristic point is a point on a sub-interval $(a+k, a+k+1)$ of the coordinate system in which $u_i(x)$ has a zero, and in fact an odd number of zeros if it has no zero at an extremity of this interval, a zero being counted an odd or an even number of times according as the function does or does not change sign in the neighborhood of the zero. If $u_i(x)$ has a zero at one end of the interval $(a+k, a+k+1)$, it does not have a zero at the other end (since $\omega(a+k) \neq 0$) and the one zero at the end is in this case a characteristic point.

The results thus obtained may be put into the form of the following theorem:

THEOREM I. *Let $u_1(x)$ and $u_2(x)$ be a fundamental system of real-valued single-valued continuous solutions of equation* (2.1) *and let $\omega(x)$ be defined by* (2.2). *Let a be a real number not less than α for which $\omega(a) \neq 0$. Then the characteristic points x for the function $u_1(x)$ and those for the function $u_2(x)$, both with respect to a, separate each other; that is, between two consecutive characteristic points for one of these functions there is one and only one for the other function.**

* Compare the closely related theorem due to M. B. Porter, *Annals of Mathematics*, (2), 3 (1902), p. 65.

This result can be readily generalized to an extensive class of functional equations of the second order. Let us consider the substitution s,

$$x' = s_x,$$

and denote by $x' = s_x^n$ the nth power of this substitution. Let it be such that there exists an open interval I of the real axis of such sort that

$$\lim_{n=\infty} s_{x_0}^n = \beta$$

for every x_0 of I, β being an end-point of I and the limit being approached monotonically. [In the case of equation (2.1) we have $s_x \equiv x + 1$ while $\alpha < x < \infty$ is a suitable open interval I for every real value of α.] *Then consider the functional equation*

$$L(x)u(x) + M(x)u(s_x) + N(x)u(s_x^2) = 0 \tag{2.3}$$

in which the functions L, M, N are real-valued single-valued continuous functions of the real variable x on the interval I and $L(x)$ and $N(x)$ are both of one and the same sign on this interval. Suppose furthermore that s_x is such that equation (2.3) *has a fundamental system of solutions $u_1(x)$, $u_2(x)$ which are real-valued single-valued and continuous on I.*

Let a be a point of I and consider the set of constants $u_i(a)$, $u_i(s_a)$, $u_i(s_a^2)$, $\cdots$; let us interpolate them linearly into the function $\bar{u}_i(x)$ with respect to the system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, s_a, s_a^2, $\cdots$. Let the zeros of $\bar{u}_i(x)$ on the part of I which is between a and β, inclusive of a and exclusive of β, be called the *characteristic points* of $u_i(x)$ with respect to a. Then, by the same procedure as in the previous case, we have the following theorem:

THEOREM II. *Let $u_1(x)$, $u_2(x)$ be a fundamental system of real-valued single-valued continuous solutions of equation* (2.3) *and let a be a point of I for which $\omega(a) \neq 0$, where*

$$\omega(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1(s_x) & u_2(s_x) \end{vmatrix}.$$

Then the characteristic points x for the function $u_1(x)$ and those for the function $u_2(x)$, both with respect to a, separate each other.

If we take $s_x \equiv qx$ where q is a real number greater [less] than unity, then a suitable interval I is that defined by the inequalities $\alpha < x < \infty$ [$\alpha > x > 0$] or that defined by the inequalities $-\alpha > x > -\infty$ [$-\alpha < x < 0$], where α is any positive number. We have therefore an interesting special case of the foregoing theorem applicable to q-difference equations of the form

$$L(x)u(x) + M(x)u(qx) + N(x)u(q^2x) = 0.$$

Each one of a great variety of functional equations finds a similar place here.

Let us next consider the analogous results for homogeneous linear

differential equations. They are classic for equations of the second order. Hence we may confine our present attention to equations of order n where $n > 2$. Such an equation we write in the general form

$$u^{(n)} + p_1 u^{(n-1)} + \cdots + p_{n-1} u' + p_n u = 0, \tag{2.4}$$

where the superscripts refer to differentiation and where the coefficients p are real-valued single-valued continuous functions of the real variable x on the interval (ab). Since this involves an n-fold infinitude of solutions we shall require boundary conditions to restrict the permissible solutions to a two-fold infinitude so as to bring the present problem into the closest possible analogy with that involved in theorem I of § 1 and so that the new theorem shall indeed be a limiting case of that theorem. We shall suppose that these conditions are of the form

$$\sum_{j=1}^{\nu} \int_a^b L_{ij}(u) d\psi_{ij}(x) = 0, i = 1, 2, \cdots, n-2, \quad \nu = \text{positive integer}, \tag{2.5}$$

the integrals being taken in the sense of Stieltjes, the functions $\psi_{ij}(x)$ being functions of bounded variation on (ab), and the $L_{ij}(u)$ denoting homogeneous linear differential expressions in u of order not greater than $n-1$ (the case when some $L_{ij}(u)$ are of the form $g_{ij}u$ being included by convention as differential expressions of order zero). We assume that the conditions are so chosen that (2.4) has two and just two linearly independent solutions satisfying (2.5). This is the case, for instance, when the conditions reduce to the following: $u(a)=0$, $u'(a)=0$, $\cdots$, $u^{(n-3)}(a)=0$.

Let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be a fundamental system of solutions of (2.4). Define the constants λ by means of the relations

$$\sum_{k=1}^{\nu} \int_a^b L_{ik}(\bar{u}_j) d\psi_{ik}(x) = \lambda_{ij}, \quad \begin{matrix} i = 1, 2, \cdots, n-2, \\ j = 1, 2, \cdots, n. \end{matrix} \tag{2.6}$$

Let u be a solution of (2.4) and write it in the form

$$u = c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_n\bar{u}_n, \tag{2.7}$$

where $c_1, c_2, \cdots, c_n$ are constants. The conditions to be met in order that u shall also satisfy conditions (2.5) may now be written in the form

$$\lambda_{i1}c_1 + \lambda_{i2}c_2 + \cdots + \lambda_{in}c_n = 0, \qquad i = 1, 2, \cdots, n-2. \tag{2.8}$$

The required condition that (2.4) and (2.5) shall have just two linearly independent simultaneous solutions now reduces to the condition that system (2.8), considered as a system for determining the coefficients $c_1, c_2, \cdots, c_n$, shall have just two linearly independent solutions; and for this it is a necessary and sufficient condition that the matrix $||\lambda_{ij}||$ of coefficients λ shall be of rank $n-2$. Thus we have a necessary and sufficient condition

that the problem (2.4), (2.5) shall have two and just two linearly independent solutions.

If we write

$$u_1(x) = c_{11}\bar{u}_1 + c_{12}\bar{u}_2 + \cdots + c_{1n}\bar{u}_n,$$
$$u_2(x) = c_{21}\bar{u}_1 + c_{22}\bar{u}_2 + \cdots + c_{2n}\bar{u}_n,$$

where u_1 and u_2 are two linearly independent real solutions of (2.4), (2.5) and the c's are constants, then we have

$$\lambda_{i1}c_{11} + \lambda_{i2}c_{12} + \cdots + \lambda_{in}c_{1n} = 0, \qquad i = 1, 2, \cdots, n-2,$$
$$\lambda_{i1}c_{21} + \lambda_{i2}c_{22} + \cdots + \lambda_{in}c_{2n} = 0, \qquad i = 1, 2, \cdots, n-2.$$

It may be shown that a constant C exists, different from zero, such that

$$\begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix} = CM_{ij},$$

where M_{ij} is the algebraic complement of

$$\begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}_i' & \bar{u}_j' \end{vmatrix}$$

in the expansion of the determinant

$$D(x) \equiv \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2,1} & \lambda_{n-2,2} & \cdots & \lambda_{n-2,n} \end{vmatrix}.$$

Now if we write

$$\omega(x) \equiv \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$$

we have

$$\omega(x) \equiv \begin{vmatrix} c_{11}\bar{u}_1 + \cdots + c_{1n}\bar{u}_n & c_{21}\bar{u}_1 + \cdots + c_{2n}\bar{u}_n \\ c_{11}\bar{u}_1' + \cdots + c_{1n}\bar{u}_n' & c_{21}\bar{u}_1' + \cdots + c_{2n}\bar{u}_n' \end{vmatrix}$$
$$\equiv \sum_{\substack{i,j=1 \\ i<j}}^{n} \begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix} \cdot \begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}_i' & \bar{u}_j' \end{vmatrix}$$
$$\equiv \sum_{\substack{i,j=1 \\ i<j}}^{n} CM_{ij} \begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}_i' & \bar{u}_j' \end{vmatrix} \equiv CD(x).$$

From this it follows that the zeros of $\omega(x)$ on (ab) are the same as those of $D(x)$ on (ab).*

* This result can be generalized to the case in which the range of i in (2·5) is over the set 1, 2, $\cdots$, k, where k is any one of the numbers 1, 2, $\cdots$, $n-1$, these conditions being so chosen that the number of linearly independent solutions of (2·4) subject to the new conditions is $n-k$. If such linearly independent solutions are denoted by u_1, u_2, $\cdots$, u_{n-k} and if constants λ are defined as before by means of a fundamental system of solutions $\bar{u}_1$, $\bar{u}_2$, $\cdots$, $\bar{u}_n$ of (2·4), then for a suitable non-zero constant C we have the identity

Let us consider any second fundamental system of solutions of (2.4). The functions in it can be represented in the form

$$\gamma_{i1}\bar{u}_1 + \gamma_{i2}\bar{u}_2 + \cdots + \gamma_{in}\bar{u}_n, \qquad i = 1, 2, \cdots, n,$$

where the γ's are constants such that the determinant $|\gamma_{ij}|$ is different from zero. If we form the determinant $D_1(x)$ of this new system of solutions, corresponding to $D(x)$ for the former system, it is easy to show by means of the multiplication theorem for determinants that we have $D_1(x) = |\gamma_{ij}|D(x)$. Hence the zeros of $D(x)$ in (ab) are independent of the fundamental system of solutions employed. The points a, b and these zeros therefore define a division of the interval (ab) into sub-intervals $I_1, I_2, I_3, \cdots$ such that $D(x)$ vanishes at the extremities of these segments other than the points a and b and does not vanish in the interior of any of these segments; and this division of (ab) into sub-intervals depends solely on the equation (2.4) and the boundary conditions (2.5).

We may now readily prove the following theorem:

THEOREM III. *If u_1 and u_2 are any two linearly independent real solutions of* (2.4), (2.5), *then between any two consecutive zeros of one of these solutions in the interior of one of the intervals $I_1, I_2, I_3, \cdots$ lies one and only one zero of the other solution.*

Since $\omega(x) \equiv CD(x)$ the determinant $\omega(x)$ is of one sign in the interior of an interval I_k. Without loss of generality we may take it to be positive in the interior of I_k; doing this we have

$$u_1u_2' - u_1'u_2 > 0 \tag{2.9}$$

within I_k. Let α and β $(\alpha < \beta)$ be two consecutive zeros of u_1 on I_k. We may (and we will) take u_1 to be positive in the interior of the interval $(\alpha\beta)$, since if it were negative we could replace u_1 and u_2 by $-u_1$ and $-u_2$ without disturbing any other assumed properties or relations. Then $u_1'(\alpha) > 0$ and $u_1'(\beta) < 0$. Then, since $u_1(\alpha) = 0 = u_1(\beta)$, it follows from (2.9) that $u_2(\alpha) < 0$ and $u_2(\beta) > 0$. Hence there is one zero of $u_2(x)$ between two consecutive zeros of $u_1(x)$ in the interior of I_k. There cannot be more than one, since the result just obtained can be used to show that a zero of $u_1(x)$

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_{n-k} \\ u_1' & u_2' & \cdots & u_{n-k}' \\ \cdots & \cdots & \cdots & \cdots \\ u_1^{(n-k-1)} & u_2^{(n-k-1)} & \cdots & u_{n-k}^{(n-k-1)} \end{vmatrix} \equiv C \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \cdots & \cdots & \cdots & \cdots \\ \bar{u}_1^{(n-k-1)} & \bar{u}_2^{(n-k-1)} & \cdots & \bar{u}_n^{(n-k-1)} \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{kn} \end{vmatrix},$$

It is possible to use this result to obtain certain generalizations of theorem III and of the consequences which result from it; but when $k < n - 2$ the results do not maintain an elegance comparable with that when $k = n - 2$.

lies between any two consecutive zeros of $u_2(x)$ on I_k. Hence the theorem is established.

Let us now consider the problem of applying the result in the foregoing theorem to the case of any two linearly independent solutions of (2.4) without reference to any preassigned boundary conditions. Let u_1 and u_2 denote any two linearly independent solutions of (2.4). Holding these solutions fixed let us consider them and the differential equation in connection with certain *associated boundary conditions* of the form (2.5). By conditions *associated* with the given differential equation and given solutions we shall mean any $n - 2$ conditions capable of expression in the form (2.5) and such that the solutions of (2.4) and the determined conditions (2.5) are those functions and those alone which are linearly dependent upon u_1 and u_2.

As a simple example of such associated boundary conditions we have those determined as follows: Let the initial constants for u_1 and u_2 at a point $x = \alpha$ of (ab) be

$$u_i^{(k)}(\alpha) = \rho_{ik}, \qquad k = 0, 1, 2, \cdots, n-1, \qquad i = 1, 2.$$

Let the coefficients σ_{ij} be so chosen that the equations

$$\sigma_{i0}u(\alpha) + \sigma_{i1}u'(\alpha) + \cdots + \sigma_{i,\,n-1}u^{(n-1)}(\alpha) = 0, \qquad i = 1, 2, \cdots, n-2 \tag{2.10}$$

have those solutions and those alone which may be written in the form

$$u^{(k)}(\alpha) = a_1\rho_{1k} + a_2\rho_{2k}, \qquad k = 0, 1, \cdots, n-1,$$

where a_1 and a_2 are arbitrary constants. Then the solutions of (2.4) and (2.10) are those functions and those alone which are linearly dependent upon u_1 and u_2. Such conditions (2.10) afford a special form of boundary conditions to be associated with the equation (2.4) and the given linearly independent solutions u_1 and u_2.

By means of the so determined boundary conditions (2.5) and a fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ of solutions of (2.4) we may define a determinant $D(x)$, as in the earlier treatment when the conditions (2.5) were preassigned. Let the zeros of $D(x)$ on (ab) be called the *special points** of (ab) with respect to the given solutions u_1 and u_2 and the named associated boundary conditions. These special points are independent of the particular fundamental system of solutions used in defining them. By aid of the foregoing theorem we now have readily the following theorem:

THEOREM IV. *If u_1 and u_2 are any two linearly independent solutions of* (2.4), *then between any two consecutive zeros α, β of u_1 on (ab) there is one*

* It is clear that the special points are not altered if u_1 and u_2 are replaced by any two linearly independent functions which are linearly dependent upon them.

and just one zero of u_2 or there is on the interval $(\alpha\beta)$ a special point of (ab) with respect to the given solutions u_1 and u_2 and any set of boundary conditions associated with (2.4) *and the given solutions u_1 and u_2 in the way indicated.*

As a very simple example illustrating the first of the two foregoing theorems let us consider the equation and condition

$$u'''(x) + u'(x) = 0, \qquad u''(0) + u(0) = 0.$$

A fundamental system of solutions of the differential equation is 1, $\sin x$, $\cos x$. The determinant $D(x)$ formed with these is now

$$D(x) \equiv \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 1 & 0 & 0 \end{vmatrix} \equiv 1.$$

Hence there are no special points. Hence any two linearly independent solutions of the given differential equation satisfying the given boundary conditions have together the property that their zeros separate each other.

In illustration of the last foregoing theorem, let us consider the two solutions

$$u_1 = a + \sin x, \qquad u_2 = \cos x$$

of the same differential equation $u'''(x) + u'(x) = 0$, where a is a given real constant. We have

$$u_1(0) = a, \quad u_1'(0) = 1, \quad u_1''(0) = 0; \quad u_2(0) = 1, \quad u_2'(0) = 0, \quad u_2''(0) = -1.$$

As a single associated boundary condition relevant in this case we may take

$$u(0) - au'(0) + u''(0) = 0.$$

Then with the fundamental system 1, $\sin x$, $\cos x$ of solutions we find that $D(x)$ has the value

$$D(x) \equiv \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 1 & -a & 0 \end{vmatrix} \equiv -(a \sin x + 1).$$

Then the corresponding special points are the zeros of $a \sin x + 1$. If $|a| < 1$ they are absent and the zeros of u_1 and u_2 separate each other. If $|a| \geqq 1$ the special points interfere with this characteristic relative distribution of the zeros of u_1 and u_2. If $|a| > 1$ it is clear that u_1 has no zeros so that in this case at least one special point must occur on every interval containing two zeros of u_2.

We have assumed that the differential equation (2.4) is of order greater than 2. It is clear that a similar and in fact much simpler analysis of the same general sort as that given above is applicable to the case when $n = 2$

and that the principal theorem which thus results is the classic theorem of Sturm. The two foregoing theorems may be considered as the natural extension of this in one important direction. Some elements of elegance are absent from the more general results, a fact which is inevitable as the simplest considerations show and as is implicitly apparent from the examples just given. The present general results afford fairly definite information as to the way in which the situation is complicated when $n > 2$.

Let $s_n(x)$ denote either $x + n$ or $q^n x$, where q is a real number greater than unity, and let us consider the functional equation of order n greater than 2,*

$$(2.11)\quad u\{s_n(x)\} + p_1 u\{s_{n-1}(x)\} + \cdots + p_{n-1} u\{s_1(x)\} + p_n u\{x\} = 0,$$

in which the coefficients $p_1, p_2, \cdots, p_n$ are real-valued single-valued continuous functions of the real variable x for $x \geqq \alpha$ (α being positive for the case of the q-difference equation). Let $u_1(x)$ and $u_2(x)$ be two real-valued single-valued continuous solutions of (2.11) which are linearly independent with respect to periodic multipliers $P(x)$ such that $P\{s_1(x)\} \equiv P(x)$. Let a be a point such that $a \geqq \alpha$ and $\omega(a) \neq 0$, where

$$\omega(x) \equiv \begin{vmatrix} u_1\{x\} & u_2(x) \\ u_1\{s_1(x)\} & u_2\{s_1(x)\} \end{vmatrix}.$$

In connection with these solutions we consider *associated boundary conditions* which are capable of representation by means of Stieltjes integrals in the form

$$(2.12)\qquad \int_a^b u\, d\psi_i(x) = 0, \qquad i = 1, 2, \cdots, n-2,$$

where b is any conveniently chosen number greater than a and the $\psi_i(x)$ are functions of bounded variation in (ab), these boundary conditions being selected in such a way that the problem (2.11), (2.12) has for real-valued single-valued continuous solutions those functions and those alone for which it is true that

$$(2.13)\quad u\{s_k(a)\} = c_1 u_1\{s_k(a)\} + c_2 u_2\{s_k(a)\}, \qquad k = 0, 1, 2, \cdots,$$

c_1 and c_2 being arbitrary constants. A particular set of such boundary conditions may be written in the form

$$\sigma_{i0} u\{a\} + \sigma_{i1} u\{s_1(a)\} + \cdots + \sigma_{i,\,n-1} u\{s_{n-1}(a)\} = 0,$$
$$i = 1, 2, \cdots, n-2,$$

* The results in the remainder of this section can be extended without difficulty to a class of functional equations obtained from those treated here by replacing $s_n(x)$ by s_x^n where $x' = s_x^n$ is the nth power of the substitution $x' = s_x$ treated in connection with the second theorem of this section. Moreover, in the case of a q-difference equation in which q is positive and less than unity we may employ the transformation $x = 1/t$, $u(x) = v(t)$ and so obtain a new q-difference equation included in the class treated in the text. Similarly, on replacing x by $-t$ one changes from a q-difference equation with negative q to one with positive q.

where the σ_{ij} are so chosen that this algebraic system has those solutions and those alone which may be written in the form of the first n equations (2.13), c_1 and c_2 being arbitrary constants.

Let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be a fundamental system of solutions of (2.11) and let constants λ be defined by means of the relations

$$\int_a^b \bar{u}_j d\psi_i(x) = \lambda_{ij}, \qquad \begin{matrix} i = 1, 2, \cdots, n-2, \\ j = 1, 2, \cdots, n. \end{matrix} \tag{2.14}$$

Then if we write

$$D(x) \equiv \begin{vmatrix} \bar{u}\{x\} & \bar{u}_2\{x\} & \cdots & \bar{u}_n(x) \\ \bar{u}_1\{s_1(x)\} & \bar{u}_2\{s_1(x)\} & \cdots & \bar{u}_n\{s_1(x)\} \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \lambda_{n-2,\,1} & \lambda_{n-2,\,2} & \cdots & \lambda_{n-2,\,n} \end{vmatrix},$$

we may readily show that $\omega\{s_k(a)\} = CD\{s_k(a)\}$, where C is independent of k, the method of proof being similar to that by which the relation $\omega(x) \equiv CD(x)$ was proved for the foregoing problem in differential equations.

By a *characteristic point* of any function $v(x)$ with respect to a we shall mean a zero (to the right of or at a) of the function $\bar{v}(x)$ obtained by linear interpolation from the constants $v\{a\}$, $v\{s_1(a)\}$, $v\{s_2(a)\}$, $\cdots$ with respect to a set of axes obtained by drawing perpendiculars to the x-axis through the points a, $s_1(a)$, $s_2(a)$, $\cdots$. The characteristic points of $D(x)$ with respect to a we shall call the *special points* for the solutions $u_1(x)$ and $u_2(x)$ with respect to the boundary conditions (2.12). They are clearly independent of the choice of fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ used in defining them.

The quantities $\omega\{s_k(a)\}$ are all of one sign if the variation of k is so restricted that all the points $s_k(a)$ lie on a single interval containing no special points. Then through use of the method employed in proving the first two theorems in this section we are led easily to the following theorem:

THEOREM V. *If u_1 and u_2 are any two real-valued single-valued continuous solutions of* (2.11) *which are independent, then between any two consecutive characteristic points α, β for u_1 with respect to a point a for which $\omega(a) \neq 0$ there is one and just one characteristic point for u_2 or there is on the interval $(\alpha\beta)$ a special point with respect to the given solutions u_1 and u_2 and any set of boundary conditions associated with* (2.11) *and the given solutions u_1 and u_2 in the way indicated.*

By taking $s_k(x) \equiv x + k$, we have in this theorem the extension, to equations of order greater than two, of the results stated for a second order difference equation in the first theorem of this section; and by taking

$s_k(x) \equiv q^k x$, $q > 1$, we have a corresponding theorem for q-difference equations.

3. *Transcendental Comparison Theorems.*—The results stated in theorem II of § 1 and the following paragraphs have as limiting cases certain of the classic theorems of comparison of Sturm for homogeneous linear differential equations of the second order; and they were indeed suggested by these Sturmian theorems. The latter, with considerable loss of elegance, have been extended to homogeneous linear differential equations of general order k.* It is possible to extend the algebraic results of the latter part of § 1 to the analogous algebraic case, namely, the case of algebraic systems with k linearly independent solutions; but the results lack (in some respects) the desired elegance. From them one may in turn obtain corresponding properties of a certain class of functional equations. We content ourselves with giving these results for the most interesting case, namely, that in which the equations have just two independent solutions.

Let us consider the functional equations

$$u(s_x^2) + \varphi(x)u(s_x) + u(x) = 0, \tag{3.1}$$

$$v(s_x^2) + \psi(x)v(s_x) + v(x) = 0, \tag{3.2}$$

in which $\varphi(x)$ and $\psi(x)$ are real-valued single-valued continuous functions of the real variable x on the interior of the interval I, where s_x, s_x^2, I are defined as in the paragraph following theorem I of § 2. If a is an interior point of the interval I, we define the *characteristic points* of a function $t(x)$ with respect to a to be the zeros on the interval $a \leqq x < \beta$ (or $a \geqq x > \beta$) of the function $t(x)$ derived from the constants $t(a)$, $t(s_a)$, $t(s_a^2)$, $\cdots$ by linear interpolation with respect to the system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, s_a, s_a^2, $\cdots$.

From theorem II of § 1 we have at once the following theorem:

THEOREM I. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. If $u(x)$ has two consecutive characteristic points with respect to a on the μth and $(m+1)$th intervals $(\mu < m)$ of the set of intervals whose end-points are the consecutive pairs of the sequence $a, s_a, s_a^2, \cdots$, then $v(x)$ has a characteristic point between these two characteristic points of u provided that either*

(a) $\varphi(x) \leqq \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, the equality sign not holding for all these end-points; or,

(b) $\varphi(x) = \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive and the two sets of constants,

$$u(s_a^{\mu-1}),\ u(s_a^{\mu}),\ \cdots,\ u(s_a^{m-1}); \qquad v(s_a^{\mu-1}),\ v(s_a^{\mu}),\ \cdots,\ v(s_a^{m-1}),$$

are linearly independent.

* *Annals of Mathematics* (2), 19 (1918): 159–171.

This theorem affords certain properties of comparison for the relative distribution of the zeros of $u(x)$ and $v(x)$. Other related properties may be obtained similarly from the italicized results following theorem II of § 1; we state them without further elaboration of the proof.

THEOREM II. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. Let us suppose that* $u(a) \neq 0$, $v(a) \neq 0$, $u(s_a)/u(a) > v(s_a)/v(a)$; *and that* $\varphi(x) \leqq \psi(x)$ *for* $x = a, s_a, s_a^2, \cdots, s_a^{\nu-1}$. *If $u(x)$ has k characteristic points on the ν intervals whose end-points are the consecutive pairs of the sequence* $a, s_a, s_a, \cdots, s_a^{\nu}$, *then $v(x)$ has at least k characteristic points on these intervals; and the jth of these characteristic points (counted from a towards s_a^{ν}) of $v(x)$ is nearer to a than the jth characteristic point of $u(x)$.*

THEOREM III. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. Let* $u(a), v(a), u(s_a^k), v(s_a^k)$ *be all different from zero and let* $u(s_a)/u(a) > v(s_a)/v(a)$. *Let $u(x)$ and $v(x)$ have the same number (which may be zero) of characteristic points on the k intervals whose end-points are the consecutive pairs of the sequence* $a, s_a, s_a^2, \cdots, s_a^k$. *Then we have*

$$\frac{u(s_a^{k+1})}{u(s_a^k)} > \frac{v(s_a^{k+1})}{v(s_a^k)},$$

provided that $\varphi(x) \leqq \psi(x)$ *for* $x = a, s_a, \cdots, s_a^{k-1}$.

Since the point a (in each of these theorems) has a great range of variation, the stated results give a large measure of information about the relative distribution of the zeros of $u(x)$ and $v(x)$ even though the arbitrary elements of the solutions of (3.1) and (3.2) are functions (rather than constants) restricted merely by a relation of the form $p(s_x) = p(x)$ and certain considerations of reality and continuity. They have their greatest use in yielding information concerning the solutions of a given equation (3.2) by comparing them with some simple functions which are known to be solutions of equation (3.1) with appropriate determination of the $\varphi(x)$.

4. *A Generalization of Boundary Conditions for Expansion Problems.*—By means of the differential expressions

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

$$L_1(u) \equiv g_m \frac{d^m u}{dx^m} + \cdots + g_1 \frac{du}{dx} + g_0 u, \qquad m < n,$$

in which the coefficients l_k, g_k with their derivatives of all orders are real-valued single-valued continuous functions of x in an interval (ab) and l_n is positive throughout (ab) while g_m does not vanish in (ab) [and hence is

either positive or negative throughout (ab)], we define the differential equation

$$L(u) + \lambda L_1(u) = 0 \tag{4.1}$$

and its adjoint

$$M(v) + \lambda M_1(v) = 0, \tag{4.2}$$

where $M(v)$ and $M_1(v)$ are the adjoints of $L(u)$ and $L_1(u)$, respectively. We have classic identities of the form

$$vL(u) - uM(v) \equiv \frac{d}{dx}\{R(u, v)\}, \quad vL_1(u) - uM_1(v) \equiv \frac{d}{dx}\{R_1(u, v)\}. \tag{4.3}$$

The quantity

$$[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b}$$

is a bilinear form in the two sets of variables

$$u^{(n-1)}(a),\ u^{(n-2)}(a),\ \cdots,\ u'(a),\ u(a),\ u^{(n-1)}(b),\ \cdots,\ u'(b),\ u(b)$$

and

$$v(a),\ v'(a),\ \cdots,\ v^{(n-2)}(a),\ v^{(n-1)}(a),\ v(b),\ v'(b),\ \cdots,\ v^{(n-1)}(b).$$

If this form is arranged in a square array whose columns contain the u's in the order written and whose rows contain the v's in the order written, then the matrix has the following properties: every element below the main diagonal is zero; every element in the upper right-hand fourth of the matrix is zero, the division being made by horizontal and vertical lines; the determinant of the matrix has the positive value $\{l_n(a)l_n(b)\}^n$.

This bilinear expression may be written in an infinite number of ways in the form

$$[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} \{U_{1i}(u) + \lambda U_{2i}(u)\}\{V_{1i}(v) + \lambda V_{2i}(v)\}, \tag{4.4}$$

where the U_{1i}, U_{2i}, V_{1i}, V_{2i} are linear homogeneous functions with constant coefficients, the first two in the variables u and the last two in the variables v, such that the determinant of the linear forms $U_{1i}(u) + \lambda U_{2i}(u)$ in the variables u is independent of λ and different from zero and such that the determinant of the linear forms $V_{1i}(v) + \lambda V_{2i}(v)$ in the variables v has the same properties. Then with (4.1) we associate the boundary conditions

$$U_{1i}(u) + \lambda U_{2i}(u) = 0, \qquad i = 1, 2, \cdots, n; \tag{4.5}$$

and with the adjoint equation (4.2) the *adjoint* boundary conditions

$$V_{1i}(v) + \lambda V_{2i}(v) = 0, \qquad i = n + 1, \cdots, 2n. \tag{4.6}$$

We say that the problem (4.1), (4.5) and the problem (4.2), (4.6) are each the *adjoint* of the other.*

The following theorems may now be proved in the same way as the special cases of them are proved in the article just cited:

THEOREM I. *If for* $\lambda = \bar{\lambda}$ *a solution* $\bar{u}(x)$ *(not identically zero) of* (4.1), (4.5) *exists, then a solution* $\bar{v}(x)$ *of* (4.2), (4.6) *also exists for* $\lambda = \bar{\lambda}$; *if* $\bar{u}(x)$ *is unique (except for a constant factor),* $\bar{v}(x)$ *is also unique (except for a constant factor).*

THEOREM II. *If* $y_1, y_2, \cdots, y_n$ *are n linearly independent solutions of* (4.1) *for* $\lambda = \bar{\lambda}$, *the condition that* $\bar{\lambda}$ *is a characteristic value is that the determinant*

$$\Delta = \begin{vmatrix} U_{11}(y_1) + \bar{\lambda}U_{21}(y_1) & U_{11}(y_2) + \bar{\lambda}U_{21}(y_2) & \cdots & U_{11}(y_n) + \bar{\lambda}U_{21}(y_n) \\ U_{12}(y_1) + \bar{\lambda}U_{22}(y_1) & U_{12}(y_2) + \bar{\lambda}U_{22}(y_2) & \cdots & U_{12}(y_n) + \bar{\lambda}U_{22}(y_n) \\ \cdots & \cdots & \cdots & \cdots \\ U_{1n}(y_1) + \bar{\lambda}U_{2n}(y_1) & U_{1n}(y_2) + \bar{\lambda}U_{2n}(y_2) & \cdots & U_{1n}(y_n) + \bar{\lambda}U_{2n}(y_n) \end{vmatrix}$$

shall vanish; a characteristic value $\bar{\lambda}$ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

Here the terms *characteristic value* and *simple characteristic value* are used in the same sense as in the treatment of the special case referred to.

From the equations

$$L(u_i) + \lambda_i L_1(u_i) = 0, \qquad M(v_j) + \lambda_j M_1(v_j) = 0$$

we have

$$\{v_j[L(u_i) + \lambda_i L_1(u_i)] - u_i[M(v_j) + \lambda_i M_1(v_j)]\} + (\lambda_i - \lambda_j)u_i M_1(v_j) = 0.$$

Integrating with respect to x from a to b and simplifying by use of equations (4.3) and (4.4) and the boundary conditions

$$U_{1t}(u_i) + \lambda_i U_{2t}(u_i) = 0, \qquad t = 1, 2, \cdots, n,$$
$$V_{1t}(v_j) + \lambda_i V_{2t}(v_j) - (\lambda_i - \lambda_j)V_{2t}(v_j) = 0, \qquad t = n+1, \cdots, 2n,$$

we have

$$(\lambda_i - \lambda_j)\sum_{k=n+1}^{2n} V_{2k}(v_j)\{U_{1k}(u_i) + \lambda_i U_{2k}(u_i)\} + (\lambda_i - \lambda_j)\int_a^b u_i M_1(v_j)dx = 0.$$

From this relation and that obtained by interchanging the rôles of u and v we deduce at once the following theorem, generalizing theorem III of the preceding paper:

THEOREM III. *If* $u_i(x)$ *and* $v_j(x)$ *are solutions of* (4.1), (4.5) *and* (4.2),

* For a special case of these adjoint problems, with references to the literature, see AMERICAN JOURNAL OF MATHEMATICS, 43 (1921): 232–270.

(4.6), *respectively, the first for* $\lambda = \lambda_i$ *and the second for* $\lambda = \lambda_j$ *and if* $\lambda_i \neq \lambda_j$, *then we have*

$$\int_a^b u_i M_1(v_j)dx + \sum_{k=n+1}^{2n} V_{2k}(v_j)\{U_{1k}(u_i) + \lambda_i U_{2k}(u_i)\} = 0,$$

$$\int_a^b v_j L_1(u_i)dx + \sum_{k=1}^{n} U_{2k}(u_i)\{V_{1k}(v_j) + \lambda_j V_{2k}(v_j)\} = 0.$$

The novelty in this form of the problem is in the appearance of the parameter λ in the boundary conditions. It is clear that a corresponding generalization exists for the various boundary value and expansion problems treated in the memoir mentioned above; and that the development of the theory follows lines closely parallel to the earlier treatment.

5. *Expansion Problems for q-Difference and Integro-q-Difference Equations.*—Let us consider the adjoint systems of q-difference equations

$$(5.1) \quad u_i(qx) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda\psi_{ij})u_j(x), \qquad i = 1, 2, \cdots, n,$$

$$(5.2) \quad v_i(x) - v_i(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda\psi_{ji})v_j(qx), \qquad i = 1, 2, \cdots, n,$$

where q is a constant whose absolute value is different from unity and where the φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and have a zero there. These systems of equations possess fundamental systems*

$$u_{1j}(x),\ u_{2j}(x),\ \cdots,\ u_{nj}(x); \qquad v_{1j}(x),\ v_{2j}(x),\ \cdots,\ v_{nj}(x)$$

of solutions each function of which is analytic at infinity, say, analytic for $|x| \geqq R$, R being an appropriately chosen positive constant; moreover, the constant term in the solution $u_{ij}(x)$, and that in the solution $v_{ij}(x)$ as well, is δ_{ij} where δ_{ij} denotes unity or zero according as j is or is not equal to i. Any solution which is analytic at infinity is made up from the foregoing solutions by taking linear combinations of them with constant coefficients. We confine attention to such solutions of (5.1) and (5.2) as are analytic for $|x| \geqq R$.

If we multiply (5.1) member by member by $v_i(qx)$ and (5.2) by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$(5.3) \qquad \sum_{i=1}^{n} \delta\{u_i(x)v_i(x)\} = 0,$$

where δ denotes the operation defined by the relation $\delta f(x) \equiv f(qx) - f(x)$. Let a be a number such that $|a| \geqq R$. In (5.3) sum as to x from a to ∞,

* These existence theorems are readily proved by means of the theory of power series. There also exists a like theory in which the point zero plays the rôle played here by the point infinity.

where x runs over the values $a, qa, q^2a, \cdots$ or the values $a, q^{-1}a, q^{-2}a, \cdots$ according as $|q|$ is greater than or less than unity; thus we have

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} = 0. \tag{5.4}$$

The first member of this equation is the analogue, for the present theory, of the bilinear form in the first member of equation (4.4). It is a bilinear form in the two sets of $2n$ variables each

$$u_1(\infty),\ u_2(\infty),\ \cdots,\ u_n(\infty),\ u_1(a),\ \cdots,\ u_n(a);$$
$$v_1(\infty),\ v_2(\infty),\ \cdots,\ v_n(\infty),\ v_1(a),\ \cdots,\ v_n(a).$$

Since the bilinear form is obviously of rank $2n$ it may be written in an infinite number of ways in the normal form

$$\sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} \equiv \sum_{i=1}^{2n} U_i(u)V_i(v), \tag{5.5}$$

where the $U_i(u)[V_i(v)]$ are homogeneous linear expressions (with constant coefficients) in the variables $u[v]$. Then with our systems of q-difference equations we associate the boundary conditions

$$U_i(u) = 0, \qquad i = 1, 2, \cdots, n; \tag{5.6}$$
$$V_i(v) = 0, \qquad i = n + 1, \cdots, 2n. \tag{5.7}$$

Then relation (5.4) is satisfied in virtue of the boundary conditions alone.

THEOREM I. *If for $\lambda = \bar{\lambda}$ a solution $\bar{u}_i(x)$ (not identically zero) of* (5.1), (5.6) *exists, then a solution $\bar{v}_i(x)$ of* (5.2), (5.7) *also exists for $\lambda = \bar{\lambda}$; if the solution $\bar{u}_i(x)$ is unique (except for a constant factor), the solution $\bar{v}_i(x)$ is also unique (except for a constant factor).*

For $\lambda = \bar{\lambda}$ we have

$$U_1(\bar{u}) = U_2(\bar{u}) = \cdots = U_n(\bar{u}) = 0, \qquad U_{n+k}(\bar{u}) \neq 0,$$

the inequality holding for some k of the set $1, 2, \cdots, n$, since from the relations $U_i(\bar{u}) = 0$ for $i = 1, 2, \cdots, 2n$ it would follow that $\bar{u}_i(\infty) = 0$ for $i = 1, 2, \cdots, n$, so that $\bar{u}_i$ would be identically zero, contrary to hypothesis. In the n-fold totality of solutions of (5.2) for $\lambda = \bar{\lambda}$, there is at least one, say $\bar{v}_i(x)$, which satisfies the $n - 1$ conditions

$$V_{n+j}(\bar{v}) = 0,$$

where j runs over all the numbers of the set $1, 2, \cdots, n$ except k. But (5.4) must be satisfied when u_i and v_i are replaced by $\bar{u}_i$ and $\bar{v}_i$, respectively, since the latter are taken for the same value of λ. Thence, through aid of the identity (5.5) and the boundary conditions for $\bar{u}_i$ and $\bar{v}_i$ already verified, we have $U_{n+k}(\bar{u})V_{n+k}(\bar{v}) = 0$. Therefore $V_{n+k}(\bar{v}) = 0$, so that the solution $\bar{v}_i(x)$ satisfies all the boundary conditions (5.7).

It remains to show that $\bar{v}_i(x)$ is unique whenever $\bar{u}_i(x)$ is unique. We shall prove this by supposing that the solution $\bar{v}_i(x)$ is not unique and then showing that the solution $\bar{u}_i(x)$ can not be unique. Since $\bar{v}_i(x)$ is now supposed to be not unique, let $\bar{v}_i(x)$ and $v_i^*(x)$ be two linearly independent solutions of (5.2) and (5.7) for $\lambda = \bar{\lambda}$. Then different numbers j and k of the set $1, 2, \cdots, n$ exist such that

$$V_j(\bar{v}) \neq 0, \qquad V_k(v^*) \neq 0.$$

If then

$$V_j(\bar{v})V_k(v^*) = V_j(v^*)V_k(\bar{v}),$$

we have

$$V_j(v^*) \neq 0, \qquad V_k(\bar{v}) \neq 0,$$

so that the functions

$$v_i(x) = V_j(v^*)\bar{v}_i(x) - V_j(\bar{v})v^*(x)$$

afford a solution of (5.2), (5.7) which is not identically zero. If

$$\begin{vmatrix} V_j(\bar{v}) & V_\alpha(\bar{v}) \\ V_j(v^*) & V_\alpha(v^*) \end{vmatrix} \tag{5.8}$$

is zero for every α of the set $1, 2, \cdots, n$, then we shall have $V_i(v) = 0$ for $i = 1, 2, \cdots, 2n$, a result which is impossible since it would imply that $v_i(\infty) = 0$ $(i = 1, 2, \cdots, n)$ and hence that $v_i(x)$ is the identically zero solution (contrary to the hypothesis about the linear independence of $\bar{v}_i(x)$ and $v_i^*(x)$). Hence for some α of the set $1, 2, \cdots, n$ the determinant (5.8) is different from zero. We fix upon such an α. Now choose $u_i^*(x)$, linearly independent of $\bar{u}_i(x)$, so as to satisfy the $n - 2$ conditions

$$U_i(u^*) = 0$$

for i running over all numbers of the set $1, 2, \cdots, n$ except j and α. Then from (5.5) and the boundary conditions already satisfied we have

$$U_j(u^*)V_j(\bar{v}) - U_\alpha(u^*)V_\alpha(\bar{v}) = 0,$$
$$U_j(u^*)V_j(v^*) - U_\alpha(u^*)V_\alpha(v^*) = 0.$$

Since the determinant (5.8) is different from zero it follows from this that $U_j(u^*) = 0$, $U_\alpha(u^*) = 0$, so that $u_i^*(x)$ satisfies (5.1) and (5.6) for $\lambda = \bar{\lambda}$, contrary to the hypothesis that these equations have unique solutions. Hence follows the truth of the second statement in the foregoing theorem.

A value of λ for which the system (5.1), (5.6) [and hence the system (5.2), (5.7)] has a solution will be called a characteristic value. The characteristic value is said to be simple if the solution corresponding to it is unique (except for a constant factor).

THEOREM II. *If*

$$y_1^{(j)}(x),\ y_2^{(j)}(x),\ \cdots,\ y_n^{(j)}(x), \qquad j = 1, 2, \cdots, n,$$

are n linearly independent solutions of (5.1) *for* $\lambda = \bar{\lambda}$, *the condition that* $\bar{\lambda}$ *is a characteristic value is that the determinant*

$$\Delta = \begin{vmatrix} U_1(y^{(1)}) & U_1(y^{(2)}) & \cdots & U_1(y^{(n)}) \\ U_2(y^{(1)}) & U_2(y^{(2)}) & \cdots & U_2(y^{(n)}) \\ \cdot & \cdot & \cdot & \cdot \\ U_n(y^{(1)}) & U_n(y^{(2)}) & \cdots & U_n(y^{(n)}) \end{vmatrix}$$

shall vanish; a characteristic value $\bar{\lambda}$ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

If we take the general solution of (5.1) for $\lambda = \bar{\lambda}$ in the form

$$u_i(x) = \sum_{j=1}^{n} c_{ij} y_i^{(j)}(x), \qquad i = 1, 2, \cdots, n,$$

we see that the vanishing of Δ is a sufficient condition that the boundary conditions (5.6) may be satisfied through an appropriate choice of the constants c_{ij}. When some first minor does not vanish this choice is unique (except for a factor which is constant through the set).

THEOREM III. *If* $u_i^{(k)}(x)$ *and* $v_i^{(l)}(x)$ *are solutions of* (5.1), (5.6) *and* (5.2), (5.7), *respectively, the first for* $\lambda = \lambda_k$ *and the second for* $\lambda = \lambda_l$ *and if* $\lambda_k \neq \lambda_l$, *then*

$$\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) u_i^{(k)}(at^s) v_j^{(l)}(at^{s+1}) = 0, \tag{5.9}$$

where $t = q$ *or* q^{-1} *according as* $|q|$ *is greater than or less than unity.*

We have the systems

$$u_i^{(k)}(qx) - u_i^{(k)}(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda_k \psi_{ij}) u_j^{(k)}(x),$$

$$v_i^{(l)}(x) - v_i^{(l)}(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda_k \psi_{ji}) v_j^{(l)}(qx) - (\lambda_k - \lambda_l) \sum_{j=1}^{n} \psi_{ji}(x) v_j^{(l)}(qx);$$

multiplying the first by $v_i^{(l)}(qx)$ and the second by $-u_i^{(k)}(x)$, adding member by member, summing as to i from 1 to n and then as to x from a to infinity over the set $a, ta, t^2a, \cdots$, we have a result which (in view of the boundary conditions and the omission of the non-zero factor $\lambda_k - \lambda_l$) reduces to the relation given in the theorem.

Let us now suppose that the problem is set up so that we have the infinitude of characteristic values $\lambda_1, \lambda_2, \lambda_3, \cdots$ and corresponding solutions of the u-problem and of the v-problem; and let us suppose that the first member of (5.9) is different from zero when l and k are equal. Then if given functions $f_i(x)$ $(i = 1, 2, \cdots, n)$ have expansions in the form

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad i = 1, 2, \cdots, n,$$

the same coefficients c_k being employed for each of the functions, these coefficients are readily determined as follows: For fixed i multiply both members of the last equation by $\psi_{ji}(x)v_j^{(k)}(qx)$, sum as to i and j from 1 to n, and sum as to x over the set a, ta, t^2a, $\cdots$; employing theorem III we come through readily to the relations*

$$c_k = \frac{\sum_{s=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(at^s)f_i(at^s)v_j^{(k)}(at^{s+1})}{\sum_{s=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(at^s)u_i^{(k)}(at^s)v_j^{(k)}(at^{s+1})}, \qquad k = 1, 2, 3, \cdots.$$

Let us illustrate these fundamental expansion formulæ of the σ-difference calculus by considering a particularly simple example. We start from the equation

$$u(qx) = \left(1 + \frac{\lambda}{x}\right)u(x), \qquad |q| < 1,$$

and its adjoint

$$\left(1 + \frac{\lambda}{x}\right)v(qx) = v(x).$$

Appropriate solutions of these are the following:

$$u(x) = 1 + \sum_{k=1}^{\infty}\frac{\lambda^k x^{-k}}{(q^{-1}-1)(q^{-2}-1)\cdots(q^{-k}-1)}, \qquad v(x) = \frac{1}{u(x)}.$$

[We may also write

$$u(x) = \prod_{k=1}^{\infty}\left(1 + \frac{q^k\lambda}{x}\right),$$

thus exhibiting $u(x)$ in factored form.] The condition (5.4) now reduces to the simpler form $u(\infty)v(\infty) - u(a)v(a) = 0$. As suitable boundary conditions implying this relation we may take

$$\alpha u(\infty) - u(a) = 0, \qquad v(\infty) - \alpha v(a) = 0,$$

where α is a given non-zero constant. Since $u(\infty) = 1 = v(\infty)$ and $u(a)v(a) = 1$, these boundary conditions require merely that λ shall satisfy the relation $u(a) = \alpha$, or

$$\alpha = 1 + \sum_{k=1}^{\infty}\frac{\lambda^k a^{-k}}{(q^{-1}-1)(q^{-2}-1)\cdots(q^{-k}-1)}.$$

The roots of this transcendental equation in λ are the characteristic values of λ for this special expansion problem. If α is not an exceptional value

* If we use a single equation of order n to replace the n equations of order unity in each system, we shall come through to an expansion problem very closely related formally to that for differential equations in § 4.

for the function of λ in the second member of this equation, the number of such characteristic values is enumerably infinite, so that we have an instance of the case supposed in the preceding paragraph. The simpler expansion formulæ pertaining to the present case may be written down at once by specializing the formulæ of the preceding paragraph.

If we apply to systems (5.1) and (5.2) a limiting process which has become classic through the investigations of Volterra, we are led through to adjoint integro-q-difference equations of the form

$$u(qx, \sigma) - u(x, \sigma) = \int_a^\beta \{\varphi(x, \sigma, \tau) + \lambda\psi(x, \sigma, \tau)\} u(x, \tau) d\tau,$$
$$v(x, \sigma) - v(qx, \sigma) = \int_a^\beta \{\varphi(x, \tau, \sigma) + \lambda\psi(x, \tau, \sigma)\} v(qx, \tau) d\tau.$$

Under appropriate hypotheses the method of successive approximation yields existence theorems for equations of this sort. For them there exists also an expansion problem analogous to that which we have just treated. In the theory of this expansion problem we have the following equations analogous to (5.4) and (5.9):

$$\int_a^\beta \{u(\infty, \sigma) v(\infty, \sigma) - u(a, \sigma) v(a, \sigma)\} d\sigma = 0,$$
$$\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) u^{(k)}(at^s, \sigma) v^{(l)}(at^{s+1}, \tau) d\sigma d\tau = 0, \qquad k \neq l.$$

Then if we have an expansion of the form

$$f(x, \sigma) = \sum_{k=1}^{\infty} c_k u^{(k)}(x, \sigma),$$

the coefficients c_k have the values

$$c_k = \frac{\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) f(at^s, \sigma) v^{(k)}(at^{s+1}, \tau) d\sigma d\tau}{\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) u^{(k)}(at^s, \sigma) v^{(k)}(at^{s+1}, \tau) d\sigma d\tau}, \qquad k = 1, 2, 3, \cdots.$$

A similar expansion theory exists for a great variety of generalizations of the q-difference and integro-q-difference equations of this section. It is of importance to have a detailed analysis of the character of the functions which are representable in the form of certain of these expansions.

UNIVERSITY OF ILLINOIS,
April, 1921.

ON A THEOREM IN GENERAL ANALYSIS AND THE INTERRELATIONS OF EIGHT FUNDAMENTAL PROPERTIES OF CLASSES OF FUNCTIONS.*

By E. W. Chittenden.

Introduction.

In his "Introduction to a Form of General Analysis"† E. H. Moore defined eight properties of classes of functions:

$$L, C, D, A, \Delta, K_1, K_2, K_{12*}.‡$$

If we denote by $\mathfrak{P}$ a general class, Δ a development of $\mathfrak{P}$ (cf. § 1), $\mathfrak{M}$ a class of functions μ on $\mathfrak{P}$ to the class $\mathfrak{A}$ of real numbers; then the properties L, C, D, A are independent of the character of $\mathfrak{P}$, while the properties Δ, K_1, K_2, K_{12*} are defined in terms of the development Δ. The property K_1 is a generalization of convergence, the property K_2 of continuity, while the property K_{12*} is defined in terms of the functions which are at once convergent and continuous on $\mathfrak{P}$ relative to Δ and $\mathfrak{M}$. The eight properties are not independent but admit among others the important relation: $D\Delta K_1 K_2$ imply K_{12*}.§

A. D. Pitcher‖ investigated the interrelations of these eight properties and their negatives and found that 108 combinations are impossible, and exhibited examples showing the possibility of 145. There remained three combinations whose character was left undetermined, out of a conceivable total of 256. His investigations led him to the conclusion that the property K_{12*} is implied by the composite property $D_1\Delta K_1 K_2 B_3$, where D_1 is a dominance property weaker than D, and B_3 is defined in terms of a development Δ.¶

The developments Δ assumed by Moore and Pitcher are finite, that is, each stage consisted of a finite number of subclasses of $\mathfrak{P}$. The author

* Presented to the American Mathematical Society at Chicago, April 21, 1916.

† The New Haven Mathematical Colloquium (Yale University Press, New Haven, 1910), pp. 1–150. To avoid frequent acknowledgments the author wishes to state that the symbolism and terminology employed are due to Moore and A. D. Pitcher (citation below).

‡ The eight properties are defined in the following pages in the order in which they occur in the discussion.

§ Moore, loc. cit., p. 145, Theorem III.

‖ "Interrelations of Eight Fundamental Properties of Classes of Functions," *The Kansas University Science Bulletin*, Vol. VII, No. 1, (1913), pp. 1–67.

¶ Pitcher, loc. cit., p. 44, Theorem I.

removed this restriction and extended the theory of MOORE to the case of infinite developments, obtaining a theorem of which the two theorems of MOORE and PITCHER mentioned are instances.*

It is the purpose of the present paper to show that the hypothesis B_3 of PITCHER is non-essential and that K_{12*} is, in fact, implied by $D_1 \Delta K_1 K_2$. It results from this proposition that two of the three properties whose character with respect to existence was not determined are self contradictory (cf. § 7).

§ 1. DEFINITIONS AND NOTATION.—We are concerned with classes $\mathfrak{M}$ of real and single-valued functions which are assumed to be defined for each element (point) of a given general class (space) $\mathfrak{P}$, and with properties of such classes $\mathfrak{M}$ defined in terms of a development Δ of $\mathfrak{P}$. Denoting the class of all real numbers by $\mathfrak{A}$, we speak of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

A development Δ of a class $\mathfrak{P}$ is a system $((\mathfrak{P}^{ml}))$ of subclasses $\mathfrak{P}^{ml}$ of $\mathfrak{P}$; the indices m, l being integral valued and the range of the index l, for fixed m, dependent on the value of m. The system of all classes $\mathfrak{P}^{ml}$ for a fixed value of the index m forms stage m of the development. A stage of a development is finite if the corresponding system $(\mathfrak{P}^{ml})$ is *finite*. A development is *finite* in case every stage is finite.†

Unless the contrary is stated the developments which occur in the following discussion are assumed to be finite. There will be, for each value of m, a finite number L^m ($\geqq 0$) of classes $\mathfrak{P}^{ml}$ of stage m.

If a point p belongs to a class $\mathfrak{P}^{ml}$, it is said to be *developed* of stage m. The class of all such points is denoted by $\mathfrak{P}^m$. Its complement $\mathfrak{P} - \mathfrak{P}^m$ is represented by the symbol $\mathfrak{P}^{-m}$, denoting the class of all points undeveloped of stage m.

THE RELATION K_{pm}. If a point p is undeveloped of stage m, or some later stage (that is, stage of index $m' > m$), it is in the relation K_{pm} relative to a development Δ of the space $\mathfrak{P}$. The class of all points p in the relation K_{pm} for given m is denoted by $\mathfrak{P}_m$. The complement $\mathfrak{P}_{-m}$ of $\mathfrak{P}_m$ is a very important class, in fact, the class of all points developed of stage m and every succeeding stage. The statement that p belongs to $\mathfrak{P}_{-m}$ is represented by the symbol $\overline{K}_{pm}$.

THE RELATION $K_{p_1 p_2 m}$. Two points p_1, p_2 of $\mathfrak{P}$ are *connected* of stage m in case there is a class $\mathfrak{P}^{ml}$ which contains both. If two points p_1, p_2 are connected of stage m or some later stage, they are in the relation $K_{p_1 p_2 m}$.

THE DOMINANCE PROPERTY D_1. A class $\mathfrak{M}$ of functions has the domi-

* "Infinite Developments and the Composition Property $(K_{12}B_1)_*$ in General Analysis."

† A development is *ultimately finite* in case there is an integer m_0 such that for all values of $m \geqq m_0$, stage m of the development is finite. The theory here developed is concerned only with the ultimate character of the developments which enter and there is no loss of generality in the restriction to *finite* instead of *ultimately finite* developments.

nance property D_1 in case for every pair μ_1, μ_2 of functions of $\mathfrak{M}$ there is a constant a and a function μ of $\mathfrak{M}$ such that the inequalities,

$$|\mu_1| \leqq a|\mu|, \qquad |\mu_2| \leqq a|\mu|, \tag{1}$$

hold uniformly on $\mathfrak{P}$. [It is customary, when writing inequalities between functions which hold uniformly in the variables, to suppress the variables.]

THE PROPERTY K_2. A class $\mathfrak{M}$ has the property $K_2(K_2\mathfrak{M})$ relative to to a development Δ of the range $\mathfrak{P}$ in case for every function μ of the class $\mathfrak{M}$ there is a function μ_2 of $\mathfrak{M}$ such that *for every small positive number e there is a positive integer m_e* such that whenever two points p_1, p_2 are in the relation $K_{p_1p_2m_e}$ then

$$|\mu_{p_1} - \mu_{p_2}| \leqq e|\mu_{2p_1}|. \tag{2}$$

[Henceforth we shall use the abbreviation: *there is an m_e*, to replace the italicized part of the preceding definition. Thus the definition of K_2 might be stated: there exists μ_2 and m_e such that $K_{p_1p_2m_e}$ implies, etc.]

§ 2. SOME CONSEQUENCES OF THE PROPERTY K_2.—If we denote the oscillation of a function μ on a class $\mathfrak{P}^{ml}$ of a development Δ of the range $\mathfrak{P}$ by $\omega^{ml}(\mu)$ and the lower bound of μ on $\mathfrak{P}^{ml}$ by $B^{ml}(\mu)$, it is easy to see from inequality (2) that whenever m exceeds m_e then

$$\omega^{ml}(\mu) \leqq 2eB^{ml}(\mu_2). \tag{3}$$

It follows that if μ_2 vanishes on $\mathfrak{P}^{ml}$ $(m \geqq m_1)$ or has the lower bound zero, then μ is constant on $\mathfrak{P}^{ml}$. Therefore, unless μ vanishes identically on $\mathfrak{P}^{ml}$, μ is bounded from zero on that class. Conversely, if μ is not constant on $\mathfrak{P}^{ml}$, then μ_2 must be bounded from zero on $\mathfrak{P}^{ml}$. Consequently, if $\bar{\mu}$ is the common dominant of μ and μ_2, the function $\bar{\mu}$ is (in view of inequality (1) and the preceding remarks) bounded from zero on every class $\mathfrak{P}^{ml}$ $(m \geqq m_1)$. This result is stated in the following theorem:

If a class $\mathfrak{M}$ of functions on a general range $\mathfrak{P}$ with a development Δ has the property K_2 relative to Δ and the property D_1, then for every function μ of the class $\mathfrak{M}$ there is a function μ and an integer m_1 such that, for every $m \geqq m_1$ and class $\mathfrak{P}^{ml}$, either μ vanishes identically on $\mathfrak{P}^{ml}$ or $\bar{\mu}$ is bounded from zero on $\mathfrak{P}^{ml}$.

The classes $\mathfrak{P}^{ml}$ on which μ vanishes identically form a set which may be null but is almost denumerably infinite. Relative to the function μ, we represent by $\mathfrak{P}^{ml_0}$ the classes on which μ is identically zero and by $\mathfrak{P}^{ml_1}$ the remaining classes of the development. The system Δ_0 of all classes $\mathfrak{P}^{ml_0}$ forms a development of $\mathfrak{P}$.* Likewise the system of all classes $\mathfrak{P}^{ml_1}$ forms a development Δ_1 of $\mathfrak{P}$. In the developments Δ_0, Δ_1 it is understood that

* We shall find it convenient to say that the function μ vanishes identically on the development Δ_0.

the stages $m < m_1$ are empty. For the purposes of this paper the development Δ may be regarded as equivalent to a development $\Delta_0 + \Delta_1$.

We understand by a *representative* system $((r^{ml}))$ of a development a system of points r^{ml}, where r^{ml} is an element of the class $\mathfrak{P}^{ml}$. If a class $\mathfrak{P}^{ml}$ is null, it will not have a representative element.

Let $\mathfrak{M}$ be a class with the property K_2, and suppose that p_1, p_2 are common elements of a class $\mathfrak{P}^{ml}$ of a stage $m \geqq m_e$ (effective in inequality (2)). Then if we replace p_1 by r^{ml} and let p_2 represent any element p of $\mathfrak{P}^{ml}$, we may write

$$|\mu_p| \leqq e|\mu_{2r^{ml}}| + |\mu_{r^{ml}}| .$$

If we set $e = 1$ and let a_m be the greatest value of $|\mu_{2r^{me}}| + |\mu_{pme}|$ for all points r^{ml} representative of classes of stage m, then we have for all points of $\mathfrak{P}^m$ $(m \geqq m_1)$

$$|\mu| \leqq a_m. \tag{4}$$

It should be noted here that the value of m_1 depends on μ.

§ 3. THE DEVELOPMENTAL PROPERTY Δ.—By definition, a class $\mathfrak{M}$ with the developmental property Δ contains a system $\mathfrak{D} = ((\delta^{ml}))$ of functions subject to two conditions of which we need consider only the first at this time: (1a) there is an m_e such that, for all $m \geqq m_e$,

$$\Big|\sum_g{}' \delta_p^{mg} - 1\Big| \leqq e, \qquad \Big|\sum_g{}' |\delta_p^{mg}| - 1\Big| \leqq e, \tag{5}$$

where g denotes a value of the index l for which p belongs to $\mathfrak{P}^{ml}$, and the prime on the summation sign denotes that the summation is restricted to existent classes of stage m. The condition applies only in case p is *developed* of stage m.

Let $\mathfrak{M}$ be a class with the properties D_1, Δ. Since the development Δ is finite there are but a finite number of developmental functions δ^{ml} of each stage m. Hence from D_1, there is a function μ_m such that, for all indices l of stage m,

$$|\delta_p^{ml}| \leqq |\mu_m| . \tag{6}$$

If we give e in inequalities (5) the value $\frac{1}{2}$, then for every point p developed of stage m there is at least one index l for which

$$|\delta_p^{ml}| \geqq \frac{1}{2\mathrm{L}^m}, \tag{7}$$

where L^m is the number of classes $\mathfrak{P}^{ml}$ of stage m of Δ.

Therefore, the function μ_m is bounded from zero on $\mathfrak{P}^m$; in fact, for every point p developed of a stage m $(m \geqq m_1$, effective in inequalities

(5)), we have

$$|\mu_{mp}| \geqq \frac{1}{2L^m}. \tag{8}$$

THE PROPERTY K_1. A class $\mathfrak{M}$ has the property $K_1(K_1\mathfrak{M})$ relative to a development Δ in case for every function μ of $\mathfrak{M}$ there is a function μ_1 of $\mathfrak{M}$ and an m_e such that K_{pm_e} implies

$$|\mu_p| \leqq e|\mu_{1p}|. \tag{9}$$

THE RELATION K_{pmm_1}. If p is a point in the relation $K_{pp_1m_1}$ with a point p_1 of $\mathfrak{P}_{-m}$ (§ 1), then p is in the relation K_{pmm_1}. The class of all such points is the class $\mathfrak{P}_{mm_1}$. It is evident that if m_1 is not less than m, $\mathfrak{P}_{mm_1} \geqq \mathfrak{P}_{-m}$.

If the class $\mathfrak{M}$ has the composite property $D_1\Delta K_1K_2$, there is for every integer m an integer $m_1 \geqq m$ such that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$.

Suppose that the inclusion $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$ fails for every value of $m_1 > m$. Then there must exist a sequence $\{p_m\}$ of points such that for every m_1 there is a point q_{m_1} of $\mathfrak{P}_{-m}$ in the relation $K_{p_{m_1}q_{m_1}m_1}$, while p_{m_1} is not in $\mathfrak{P}_{-m_1}$, that is, the relation $K_{p_{m_1}m_1}$ holds.

Let μ be any function of the class $\mathfrak{M}$. Then, because $\mathfrak{M}$ has the property K_2, we have μ_2 and m_e such that K_{pqm_e} implies

$$|\mu_p - \mu_q| \leqq e|\mu_{2q}|. \tag{2'}$$

Since $\mathfrak{P}_{-m}$ belongs to $\mathfrak{P}^{m_1}$ for every value of $m_1 > m$, it follows from (4) that μ and μ_2 are limited on $\mathfrak{P}_{-m}$. Therefore μ and μ_2 are limited on the sequence $\{q_m\}$. Hence μ and μ_2 are limited on the sequence $\{p_m\}$.

But, from the hypothesis that the class $\mathfrak{M}$ has the property K_1, there is a function μ_1 and an integer m_e such that K_{pm_e} implies

$$|\mu_p| \leqq e|\mu_{1p}|.$$

Since μ_1 must be bounded on the sequence $\{p_m\}$ it follows from the above inequality and the hypothesis on the sequence $\{p_m\}$ that $|\mu_{p_m}|$ tends toward zero with m.

However, the class $\mathfrak{M}$ contains a function μ bounded from zero on $\mathfrak{P}^{m_1}$ for some value of $m_1 > m$. This function μ must therefore exceed some number $a_0 > 0$ at all points of $\mathfrak{P}_{-m}$ ($\leqq \mathfrak{P}^{m_1}$). But from inequality (2') we infer, for all k such that $k \geqq m_e$, that the relation $K_{p_kq_km_e}$ holds, and

$$\begin{aligned} |\mu_{p_k}| &\geqq |\mu_{q_k}| - e|\mu_{2q_k}| \\ &\geqq a_0 - eb_2, \end{aligned}$$

where b_2 denotes the least upper bound of μ_2 on $\mathfrak{P}_{-m}$. Since e is arbitrary, the sequence $|\mu_{p_k}|$ cannot tend towards zero, contrary to the result pre-

viously obtained. Hence the hypothesis that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$ fails for every $m_1 > m$ is untenable and the theorem is established.

From the results of § 2 and the present article we readily infer the following:

If a class $\mathfrak{M}$ has the composite property $D_1\Delta K_2$, every function μ of $\mathfrak{M}$ is bounded on $\mathfrak{P}_{-m}$ for every value of m. For every positive integer m there is a function μ_m bounded from zero on $\mathfrak{P}_{-m}$.

Furthermore, if $\mathfrak{M}$ has the additional property K_1, then, for every positive integer m, there is an integer $m_1 > m$ such that $\mathfrak{P}_{mm_1} \leqq \mathfrak{P}_{-m_1}$, every function μ is bounded on $\mathfrak{P}_{mm_1}$, and there is a function μ_{m_1} bounded from zero on $\mathfrak{P}_{mm_1}$.

§ 4. ON FUNCTIONS OF THE CLASS $(\mathfrak{M}'\mathfrak{M}'')_*$.—Denoting by $\mathfrak{P}'$, $\mathfrak{P}''$ two ranges which are conceptually distinct and by $\mathfrak{M}'$, $\mathfrak{M}''$ classes of functions μ', μ'' on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, we define the class $(\mathfrak{M}'\mathfrak{M}'')_*$ as the class of all functions which are the limits of sequences of linear combinations of functions of the multiplicational composite class $\mathfrak{M}'\mathfrak{M}''$, the convergence being uniform relative to a scale function from the class $\mathfrak{M}'\mathfrak{M}''$. That is, if we introduce the notation

$$\Theta_n = \sum_{j=1}^{j=n} a_{nj}\mu'_{nj}\mu''_{nj},$$

the class $(\mathfrak{M}'\mathfrak{M}'')_*$ is the class of all functions Θ of the form

$$\Theta = L_n\Theta_n,$$

where the convergence is uniform on $\mathfrak{P}'\mathfrak{P}''$ relative to some scale function $\mu'\mu''$ of the class $\mathfrak{M}'\mathfrak{M}''$. By definition, there is an n_e such that for all $n \geqq n_e$

$$|\Theta - \Theta_n| \leqq e|\mu'\mu''|,$$

uniformly on $\mathfrak{P}'\mathfrak{P}''$.

The following proposition is an immediate consequence of this inequality:

If the classes $\mathfrak{M}'$, $\mathfrak{M}''$ each have the dominance property D_1, there exist a function $\tau_1 = \tau'_1\tau''_1$ of $\mathfrak{M}'\mathfrak{M}''$ and an integer n_1 such that, for all $n \geqq n_1$, $a\tau_1$ dominates Θ, $\mu'\mu''$, and Θ_n whenever a is a sufficiently large positive constant.

Let $\Delta'_0 = ((\mathfrak{P}'^{ml_0}))$ be the system of all classes $\mathfrak{P}'^{ml_0}$ on which τ'_1 vanishes identically. Then all the functions Θ, $\mu'\mu''$, Θ_n $(n \geqq n_1)$ vanish identically on Δ_0. We recall, from § 2, that there exists a function σ' which dominates τ'_1 and is bounded from zero on each class $\mathfrak{P}'^{ml_1}$ of the development Δ'_1 whenever the class $\mathfrak{M}'$ has the properties D_1, K'_2.

The following theorem of PITCHER (loc. cit., p. 30) is of importance in the sequel:

If relative to a development Δ' of $\mathfrak{P}'$ the class $\mathfrak{M}'$ has the property K'_1, and if $\mathfrak{M}'$ and $\mathfrak{M}''$ each have the dominance property D_1, then every function Θ of $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_1\mathfrak{M}'(\mathfrak{M}'')$.

§ 5. CONDITIONS IMPLYING THAT THE CLASS $(\mathfrak{M}'\mathfrak{M}'')_*$ HAVE THE PROPERTY $K_2'\mathfrak{M}'(\mathfrak{M}'')$.—MOORE has shown that if $\mathfrak{M}'$ has the property DK_{12}', and $\mathfrak{M}''$ has the property D,* the class $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$.† PITCHER showed that the property D in the hypothesis on $\mathfrak{M}'$, can be replaced by the weaker property D_1B_3'. The property B_3' will be defined later. We shall show that the conclusion is true under the hypothesis, $\mathfrak{M}'$ has the property $D_1\Delta'K_1'K_2'$.

If a class $\mathfrak{M}'$ has the composite property $D_1\Delta'K_{12}'$, and a class $\mathfrak{M}''$ has the property D, *every function of the class $(\mathfrak{M}'\mathfrak{M}'')_*$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$.*

The proof of this theorem follows the plan of the proof of the corresponding theorem of PITCHER,‡ differing from his proof in details whose character will be indicated later.

From the theorem of MOORE stated in § 4 there exists for every function Θ of $(\mathfrak{M}'\mathfrak{M}'')_*$ a scale function $\mu_1'\mu_1''$ such that $K_{pm_e'}'$ implies:

$$|\Theta_{p'}| \leqq e|\mu_{1p'}'\mu_1''|. \tag{10}$$

Since $\mathfrak{M}'$ has the property K_2' there is a function μ_0' and integer m_e'' such that $K'_{p_1'p_2'm_e''}$ implies:

$$|\mu_{1p_1'}' - \mu_{1p_2'}'| \leqq e|\mu_{0p_1'}'|. \tag{11}$$

It readily follows from the two preceding inequalities that there is an m_e such that the simultaneous presence of the relations

$$K'_{p_1'm_e}, \qquad K'_{p_2'm_e}, \qquad K'_{p_1'p_2'm_e}$$

implies

$$|\Theta_{p_1'}| + |\Theta_{p_2'}| \leqq e|\mu_{0p_1'}'\mu_1''|. \tag{12}$$

By definition, if a class $\mathfrak{M}$ has the dominance property D, there exists for every sequence $\{\mu_m\}$ of functions of $\mathfrak{M}$ a function μ and sequence $\{a_m\}$ of positive real numbers such that for every m, $|a_m\mu| \geqq |\mu_m|$. It is at once evident that any class of functions which is limited on a range Ω and contains a function bounded from zero on Ω has the property D.

The development Δ' of $\mathfrak{P}'$ determines by an obvious reduction a development Δ_m' of the class $\mathfrak{P}'_{-m}$. From the results of §§ 2, 3 it follows that

* A class $\mathfrak{M}$ has the dominance property D in case there exists for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ a function μ of $\mathfrak{M}$ and sequence $\{a_n\}$ of positive constants such that for every n, $|\mu_n| \leqq a_n|\mu|$.

† A function θ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$ in case there is a function $\mu'\mu''$ of $\mathfrak{M}'\mathfrak{M}''$ and an m_e such that $K'_{p_1'p_2'm_e}$ implies

$$|\Theta_{p_1'} - \Theta_{p_2'}| \leqq e|\mu_{p_1'}'\mu''|.$$

A class N of functions on $\mathfrak{P}'\mathfrak{P}''$ has the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$ in case every function Θ of N has the property.

‡ Cf. PITCHER, loc. cit., § 39, pp. 40–43.

relative to $\mathfrak{P}'_{-m}$ the class $\mathfrak{M}'$ has the property D. It is easy to see that every function has the property K'_1 relative to the development Δ'_m of $\mathfrak{P}'_{-m}$, since no element of $\mathfrak{P}'_{-m}$ is undeveloped after stage m. Hence the class $\mathfrak{M}'_m$ of functions on $\mathfrak{P}'_{-m}$, obtained by considering the functions of $\mathfrak{M}$ on $\mathfrak{P}'_{-m}$ only, has the composite property DK'_{12} relative to to Δ'_m. The result of MOORE cited above applies here, and we conclude that relative to Δ'_m the function Θ has the property $K'_2\mathfrak{M}'(\mathfrak{M}'')$. That is, there exists, for every integer m, a function $\mu'_m\mu''_m$ of $\mathfrak{M}'\mathfrak{M}''$ and an integer m_{em} such that $K'_{p'_1p'_2m_{em}}$ implies (when p'_1, p'_2 are points of P'_{-m})

$$|\Theta_{p'_1} - \Theta_{p'_2}| \leqq e\,|\mu'_{m\,p_1}\mu''_m|. \tag{13}$$

We have seen in § 4 that there exist in $\mathfrak{M}'\mathfrak{M}''$ a function $\tau_1 = \tau'_1\tau''_1$ such that τ_1 dominates Θ, and that, relative to τ'_1, $\Delta' = \Delta'_0 + \Delta'_1$, where τ'_1 vanishes identically on Δ'_0, and a function σ', dominating τ'_1, which is bounded from zero on every class $\mathfrak{P}^{ml_1}$ of Δ'_1.

Since σ' is bounded from zero on the points of $\mathfrak{P}'_{-m}$ which belong to classes $\mathfrak{P}'^{ml_1}$, and since μ'_m is bounded on $\mathfrak{P}'_{-m}$, we may find a constant a'_m such that $|\mu'_m| \leqq a'_m|\sigma'|$ for all such points of $\mathfrak{P}'_{-m}$. From the property D (of the class $\mathfrak{M}''$) there is a function σ'' in $\mathfrak{M}''$ and a sequence $\{a''_m\}$ of positive constants such that, for all points of $\mathfrak{P}''$,

$$|\mu''_m| \leqq a''_m|\sigma''|.$$

Replacing $\mu'_m\mu''_m$ in inequalities (13) by $\sigma'\sigma''$ we obtain m_{em} such that $K'_{p'_1p'_2m_{em}}$ implies, whenever p'_1, p'_2 are elements of $\mathfrak{P}'_{-m}$,

$$|\Theta_{p_1} - \Theta_{p'_2}| \leqq e\,|\sigma'_{p'_1}\sigma''|. \tag{14}$$

For, if p'_1, p'_2 are connected of stage $m' \geqq m_{em}$ of Δ', they are connected of this stage in one of the two developments Δ'_0, Δ'_1. In the first case $\Theta_{p'_1} = \Theta_{p'_2} = 0$ (p''), and in the second case

$$|\mu'_{m\,p'_1}\mu''_m| \leqq a'_m a''_m\,|\sigma'_{p'_1}\sigma''|.*$$

Let $\nu = \nu'\nu''$ be a common dominant of $\mu'_1\mu''$, $\mu'_0\mu''_1$, $\sigma'\sigma''$. Then there is an integer m_e such that

$$K'_{p'_1m_e}K'_{p'_2m_e}K'_{p'_1p'_2m_e} \quad \textit{implies} \quad |\Theta_{p'_1}| + |\Theta_{p'_2}| \leqq e\,|\nu_{p'_1}|. \tag{15}$$

From a theorem of § 3, we have an integer m'_e (dependent on m_e) such that

* The inequality (14) can be established with less difficulty, in fact without the intervention of the system Δ_0, if the class $\mathfrak{M}'$ contains a function μ' bounded from zero on $\mathfrak{P}'_{-m}$ for every value of m. This fact led A. D. PITCHER to make this hypothesis which he introduced as the property B'_3. In the present proof the hypothesis Δ is introduced to provide functions bounded from zero on the classes $\mathfrak{P}_{-m}$. The hypothesis B_3 provides a single function effective on all $\mathfrak{P}_{-m}$. Such functions are needed in order to prove that the reduction of $\mathfrak{M}'$ relative to $\mathfrak{P}_{-m}$ has the dominance property D.

whenever $m \geqq m'_e$ then $\mathfrak{P}'_{m_e m} \leqq \mathfrak{P}'_{-m}$; and for $m = m'_e$ there is, because of inequality (14), an integer $m''_e > m'_e$ such that

$$\overline{\mathrm{K}}'_{p'_1 m'_e}\overline{\mathrm{K}}'_{p'_2 m_e}\mathrm{K}'_{p'_1 p'_2 m''_e} \qquad \textit{implies} \qquad |\Theta_{p'_1} - \Theta_{p'_2}| \leqq e\,|\nu_{p'_1}|. \tag{16}$$

It follows, from the inclusion of $\mathfrak{P}'_{m_e m'_e}$ in $\mathfrak{P}'_{-m'_e}$, that $\mathrm{K}'_{p_1 p_2 m_e}$ implies that both p'_1, p'_2 belong to $\mathfrak{P}_{-m'^e}$ or else neither is contained in $\mathfrak{P}'_{-m_e}$. It is now easy to see that if p'_1, p'_2 are in the relation $\mathrm{K}'_{p'_1 p'_2 m''_e}$, then one of the inequalities (15), (16) must hold. That is, there is a function μ $(= \nu)$ and positive integer m_e $(= m''_e)$ such that

$$\mathrm{K}'_{p'_1 p'_2 m_e} \qquad \textit{implies} \qquad |\Theta_{p'_1} - \Theta_{p'_2}| \leqq e\,|\mu_{p'_1}|, \tag{17}$$

which was to be proved.

§ 6. THE PROPERTY K_{12*}.—A class $\mathfrak{M}'$ has the property K'_{12*} if for every class $\mathfrak{M}''$ with the composite property LCD* the corresponding class $(\mathfrak{M}'\mathfrak{M}'')_*$ is *equal* to the class of all functions Θ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$ which are functions of $\mathfrak{M}''$ for every element p' of $\mathfrak{P}'$ and have the property $\mathrm{K}'_{12}\mathfrak{M}'(\mathfrak{M}'')$. When the class $\mathfrak{M}''$ does not enter explicitly we drop the prime and speak of the property K_{12*} of a class $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$.

MOORE demonstrated that the property $\mathrm{D}\Delta\mathrm{K}_1\mathrm{K}_2$ implies the property K_{12*}. PITCHER showed that K_{12*} is implied by $\mathrm{D}_1\Delta\mathrm{K}_1\mathrm{K}_2B_3$, where B_3 is defined in terms of the development Δ (cf. § 5). From the result of § 5 and the following theorem of MOORE, 'if $\mathfrak{M}'$ has the properties D_1, Δ', K'_1, K'_2, and $\mathfrak{M}''$ has the property LCD, every function which has the property $\mathrm{K}'_{12}\mathfrak{M}'(\mathfrak{M}'')$ and belongs to $\mathfrak{A}\mathfrak{M}''$ for every point p' of $\mathfrak{P}'$ belongs to $(\mathfrak{M}'\mathfrak{M}'')_*$,'† we have the theorem which forms the subject of this paper:

If $\mathfrak{M}$ is a class of functions on a range $\mathfrak{P}$ to the class $\mathfrak{A}$ of real numbers and has the properties D_1, Δ, K_1, K_2, *relative to a finite development Δ of $\mathfrak{P}$, then the class $\mathfrak{M}$ has the property* K_{12*}.

In view of related results of MOORE we obtain at once the further conclusion:

The class $\mathfrak{M}_*$ has the properties L, C, D_1, Δ, K_1, K_2, K_{12*}.

§ 7. ON THE INTERRELATIONS OF THE EIGHT PROPERTIES L, C, D, A, Δ, K_1, K_2, K_{12*}.—All of the eight properties, L, C, D, A, Δ, K_1, K_2, K_{12*}, have been defined previously except the property A. A class $\mathfrak{M}$ is absolute

* The property D is defined in a footnote to § 5. A class $\mathfrak{M}$ of functions is LINEAR (L) if it contains the function $a_1\mu_1 + a_2\mu_2$, whenever μ_1 and μ_2 are functions of $\mathfrak{M}$ and a_1, a_2 are real constants. A class $\mathfrak{M}$ is CLOSED (C) if it contains all functions Θ of the form

$$\Theta = \mathrm{L}_n\mu_n(\mathfrak{P};\, \mu),$$

that is, all limits of sequences of functions of $\mathfrak{M}$ converging uniformly relative to a scale function of the class $\mathfrak{M}$.

† MOORE, loc. cit., § 80, Theorem II, p. 140. The class $\mathfrak{A}\mathfrak{M}''$ is the class of all functions $a\mu''$, where a is a real number and μ'' belongs to $\mathfrak{M}''$.

(A) if it contains the absolute function $|\mu|$ of every function μ which belongs to $\mathfrak{M}$.

If a class $\mathfrak{M}$ is linear (L) and absolute (A), it has the dominance property D_1. It follows that the property $LA\Delta K_1K_2$ implies K_{12*} (§ 6). Denoting the absence of a property P by the symbol $-P$, we represent two of the three composite properties whose character was left undetermined by PITCHER as follows:

$$LC\,{-}DA\Delta K_1K_2\,{-}K_{12*}$$
$$L\,{-}C\,{-}DA\Delta K_1K_2\,{-}K_{12*}.$$

It is clear that these combinations are self contradictory.

Of the 256 possible combinations of the eight properties the results of PITCHER and the present paper show that 145 exist and 110 do not exist as composite properties of a class $\mathfrak{M}$ of functions.

The remaining combination

$$LC\,{-}D\,{-}A\Delta K_1K_2\,{-}K_{12*}$$

presents an unsolved problem. PITCHER has exhibited a number of classes of systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$ for which this composite property does not exist. From the theorem of § 6 we may add that this composite property is non-existent in any system in which the class $\mathfrak{M}$ has the property D_1.

THE UNIVERSITY OF IOWA,
AUGUST, 1920.